JUNIOR HIGH SCHOOL MATHEMATICS

BOOK II

THEODORE LINDQUIST

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BY

THEODORE LINDQUIST, PH.D.
HEAD DEPARTMENT OF MATHEMATICS
KANSAS STATE NORMAL, EMPORIA, KANSAS

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B

TO THE TEACHER

In this book, as in Junior High School Mathematics, Book I, the subject is presented in a natural psychological way. As in Book I, the treatment is also in accordance with the accepted views of a great majority of educators who have made exhaustive investigation of the ways and means available to get the most value out of mathematics in the organized junior and senior high school courses.

The work for the seventh year found in Book I has been connected by indissoluble bonds with the work of the eighth year found in this succeeding book, Book II, and the work found here will in turn be connected with the work to be found in Book III by equally close and strong connecting ties. The author, for the purpose of providing a continuous stream of consecutive work for a three-year course, has adopted the same basic method. It is best described, perhaps, by calling it without fear or favor the topical plan spiralized. Each topic is treated at sufficient length to create a lasting impression, and ever thereafter at intervals is brought up in reviews. The need of this eternal vigilance in review work, and the special demand of modern business and industry for greater facility in the handling of the four fundamental operations, is met by introducing this second book with the four fundamental operations with integers. As in Book I, these reviews are elastic, and can therefore be made to fit the needs of any particular class. Short cuts, approximations, and checks are given due im-

portance throughout the computations. The value of new associations and the application of principles studied in engaging the pupil's interests is recognized. For these reasons the approach is varied from that in Book I, and new elements are introduced, as, for instance, horizontal addition, abbreviated multiplication and division, and simplified addition.

Decimals being merely an extension of our number system to fractions, and the pupil being already familiar with the use of decimals, these are treated in connection with integers. Literal numbers introduced in Book I are now studied much more fully. The equation and formula are accepted as the chief literal mathematical tools, and work is provided as needed for extended development of these two main elements. Much of the literal work that is often given, like complicated factoring and operations with several sets of grouping signs, has been eliminated, because they occur only in very technical matters. It has also been found that only a few exercises are needed to make some principles and operations clear and to fix them in the memory. In providing exercise material this was taken into account, thus saving time and energy. Literal fractions are naturally studied with numerical fractions, and the same curtailment practised as with integers. The pupil who has completed the literal number work of this volume can successfully handle third semester algebra.

Graphs are first studied as a new mathematical language, for the comparison of quantities. Some of these sets of quantities are unrelated, as, for instance, the wheat yield in the various states for a certain year, while others are connected by a functional relationship, as the radius and area of a circle. In order to bring out this use of the graph, much attention is given to the reading of graphs as well as to their

construction. The second use of graphs is in the solution of equations. They are made the chief means of solving simultaneous quadratic equations.

The two forms of number comparisons, majorities and ratios, are treated together in the same chapter. Considerable attention is paid to the language usually met, as well as to the various forms of comparison that arise under either majorities or ratios. Among the forms studied are the simple, practical, trigonometric ratios which can be used in a multitude of interesting problems.

A chapter is devoted to that useful time-saving instrument of calculation, logarithms. They are treated as exponents, and are developed easily from the known laws of exponents. The pupil is not mystified by the meaningless words "characteristic" and "mantissa," but is given the perfectly intelligible words "whole number" and "decimal part" instead.

The tables have again been placed together in the back of the book. Any material in the tables can be found there much more easily than if scattered throughout the book. Of greater importance still is the fact that the pupil should learn how to use tables intelligently, and it is in this collective form that he will always meet with tables after his school hours are ended.

The problems and projects used in this volume as vehicles for putting operations and principles into practice are such as arise in the pupil's daily life, and have been selected from things which are within his comprehension, and which appeal to his interest. A variety of such motive material has been used in order to teach the pupil that a mathematical instrument is useful in securing a needed result, the need for which may arise under widely varying conditions. Thus and thus only will the pupil retain command of the prin-

ciples he has applied to interesting subjects and material, and will be able to use these principles to meet varying questions and problems requiring some form of mathematical statement and solution, as they arise in after life. This motive element has been drawn largely from sports, from past and contemporaneous history, from thrift, economy and social work, and from widely diversified interests associated with the development of good and useful citizenship.

Suggestions are made throughout for the application of principles and operations to local situations, and the teacher will greatly increase the value of the pupils' work by appealing to them to become interested in the particular local conditions surrounding them, and from these to select and contribute to the recitation period questions of point and of importance. While this at first may seem an extra burden for the teacher, local problems and projects will soon suggets themselves as a natural consequence of classroom and other daily activities, and will socialize the recitation, which invariably makes for better results.

The pupil should also be led to study everything in which he is interested from the quantitative standpoint. In this correlation of subjects a fund of valuable instruction will be found. Of course care should be taken, and the pupils should never be asked or permitted to pry into private affairs, such as asking each pupil to tell his father's income. Before assigning any outside work, the teacher should make certain that it can be successfully carried out by the class.

The author wishes to express his appreciation of the many valuable criticisms and suggestions that have come from students and fellow teachers; in particular to Mr. W. H. Keller and Miss Inez Morris, of the Kansas State Normal; and to Miss Lena B. Hansen, of the Enid, Oklahoma,

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THEODORE LINDQUIST.

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JUNIOR HIGH SCHOOL MATHEMATICS

BOOK II

I

HINDU-ARABIC NUMBERS

1. Decimal Counting.—Many hundred years ago counting and computing were carried on by the use of fingers, pebbles, sticks, knots in ropes, and sand strewn upon stone tablets. Counting 10 units to make 1 ten, 10 tens to make 1 hundred, and so on, has likely arisen from finger counting. We even call our number symbols 1, 2, 3, etc., digits, which is also another name for fingers. A counting system based upon tens, as is ours, is called a **decimal system.**

2. Early Number Systems.—Separate symbols were used for each number in the early number systems. In one such system 10 was $\wedge$, 20 was $\overline{\wedge}$, 100 was $\prime$, and 200 was $\cdot\prime$. Symbols written one after the other were usually added together. Thus, 50 would be $\overline{\wedge}\ \overline{\wedge}\ \wedge$. How?

3. Roman Number System.—The **Roman number system** was one of the early systems and is still used to-day for some things. In this I, V, X, L, C, D, M stand for 1, 5, 10, 50, 100, 500, 1000. A dash over any symbol multiplies it by 1000. Thus $\overline{\text{V}}$ is 5000. Any symbol **following** one of the same or of larger value is **added** to the one before it. Any symbol **preceding** one of larger value is **taken from** the one following it. In this way are made 4 (IV), 9 (IX), 40 (XL), 90 (XC), and also 400, 900, 4000, 9000. Express these last four numbers in Roman numerals.

4. Hindu-Arabic Number System.—Over 1000 years ago the Hindus, in far away India, invented our present simple number system. They used 9 numerals as we do the 1, 2, 3, etc., but these were slightly different in form. These numerals—digits—they combined into numbers based upon the decimal idea just as is done to-day. Each of the digits, 2, 4, and 8, in the number 248 has a definite value. Each digit also has a value due to its position in the number. This is called its **place value.** Thus, 482 is not the same number as 248. What other numbers can be made from the digits 2, 4, and 8? Which is the largest? Which is the smallest?

At the time that the Hindus invented their number system the Arabs were the chief merchants of the world and carried goods from one country to another. In trading with the Hindus, the Arabs learned this simple number system, which they adopted. As they traded with the people of Europe and of Africa, they brought the Hindu system of numbers into these countries. The Arabs had nothing to do with inventing this simple number system, but because they brought it into Europe it has incorrectly been called the Arabic number system.

5. Fractions in the Hindu Number System.—In 1548 Simon Stevin of Flanders invented **decimal fractions** which are an extension of the Hindu number system to fractions. In 387.42 the 4 is one-tenth of what it would be where the 7 is; the 2 is one-tenth of what it would be where the 4 is and so on. In reading numbers use **and** for the decimal point but at no other time.

EXERCISES

1. Read the following: IX; XI; XXIV; XL; LX; LXV; CL; MC; CM; CMXIV; MCMXVIII; MCMXX; MCMXIX; $\overline{\text{X}}$; $\overline{\text{L}}$.

2. Express the following in Roman numerals: 15; 20; 25; 75; 205; 410; 1905; 1050; 1910; 1918; 1924; 5000; 6000.

3. What is the value of each of the digits in 4444?

4. Separate each of the following numbers into periods of three digits and read:

243568	36534706	3781936	68432
3451793	7340560	450067	90715
12720040	10270800	15300508	1638905
640230	4830670	7620007	2074869

5. Explain the value of each digit in 347.62; in 45.73; in 3.465; in 5.307; in 34.005; in 2.4506.

6. Read the following decimals: 304.56; 2304.245; 3.4506; 30.2054; 19.0045; 0.00452; 1.0504; 305.0072; 0.00025; 1.00503; 0.02056; 5.000035; 0.030502.

7. Write the following numbers: one hundred thirty-five and two tenths; seventy-four and twenty-six hundredths; one thousand fifteen and thirty-four thousandths; ninety and one hundred eight tens-of-thousandths; forty hundredths.

6. Reading and Writing Numbers.—When writing numbers in words be careful to use the hyphen in such numbers as twenty-five, sixty-four, and so on. Use **and** only where the decimal point comes. Thus, thirty-seven dollars **and** twenty-nine cents.

Business and scientific men read such numbers as 2453.78, "twenty-four, fifty-three, point, seventy-eight"; 63.07, "sixty-three, point, oh, seven." As a telephone number, 4526 would be called "four, five, two, six."

EXERCISES

1. Write the following numbers in words: 35; 83; 70; 165; 235; 5,362; 306; 5,073; 7,004; 2,049; 8,000; 34,007.

Read the following numbers as a business or scientific man would:

2. 4573	7. 4506.34	12. 134.0561	17. 15.0045
3. 3046	8. 2745.62	13. 305.2407	18. 25.4505
4. 1908	9. 3709.36	14. 145.0045	19. 105.3462
5. 2647	10. 4325.05	15. 625.1073	20. 45.3207
6. 2893	11. 5405.26	16. 106.6372	21. 1.0562

22. Read the numbers in Exs. 2 to 6 as telephone numbers.

23. How are house numbers read? Read the numbers in Exs. 2 to 6 in this manner.

7. Addition.—Learn the addition combinations so thoroughly that $6 + 5$ and $5 + 6$ suggest 11 as quickly as d-o-g suggests dog. In adding 9 increase ten's digit by 1 and decrease unit's digit by 1. Thus, $45 + 9 = (40 + 10) + (5 - 1) = 54$. In adding 8 increase ten's digit by 1 and decrease unit's digit by 2. Thus, $65 + 8 = (60 + 10) + (5 - 2) = 73$.

EXERCISES

1. Give the sum of each digit and the one to the right; to the left. Give the sum of each digit and the one just above it; just below it.

6	7	9	5	4	7	6	8	8	6	5	7
5	8	7	9	6	8	6	9	6	7	8	4
7	6	5	6	8	9	5	4	7	8	9	7
4	7	8	5	9	4	7	8	6	5	4	6
7	8	3	6	5	8	5	9	5	7	3	8
5	4	8	9	7	4	7	3	8	6	9	5
9	5	7	4	3	8	5	9	7	8	6	8
4	7	6	9	8	7	8	4	5	6	7	4
9	3	8	5	4	8	6	9	7	5	8	3
4	9	6	7	8	5	7	4	9	6	7	8

Give the following sums as quickly as possible:

2.	45	64	37	86	74	63	64	76	34
	9	8	8	9	9	8	9	8	9
3.	35	68	24	58	52	48	73	57	65
	8	9	8	8	9	3	8	9	8
4.	49	68	78	39	27	89	28	69	56
	3	6	7	6	8	3	5	4	9

5. Find the sum of each of the columns in 1.

8. Column Addition.—Some who add rapidly pick such combinations as make 10: $2 + 3 + 5$; $7 + 3$, Others pick out digits which repeat in a column and m ply that digit by the number of times it occurs in the colu

```
 3 2 3
 4 5 7-2
 6-4-2 9-
 5 7 2 1
 9-4-3 5
 6-4 2 7-
 3 2 8-3
 9-6-1 5-
---------
4 5 4 8 2
```

A very simple scheme for rapid addition is to beg the top of the column and add until 10 or a numbe tween 10 and 20 is reached. The 10 is dropped a small mark made to indicate the fact. The unit's is added to the next digit until another mark is mad so on. The last unit's digit is written down in the The dots are then counted and this number is p over the ten's column to be added to the tens. Rei ing columns are added in the same manner.

Check by adding each column upward and downwar(Add each column at the bottom of page 4.

9. Horizontal Addition.—It is often very necessar add horizontally as well as vertically. Be careful nc add tens to units, hundreds to tens, and so on.

EXERCISES

1. Find the sum of the digits in each horizontal rov page 5.

2. Columns which need to be added horizontally vertically arise in banks, city and county treasu offices, government statistics, and in magazines. Tr find some for the class to add.

Read the numbers at the top of the next page as a l ness or a scientific man would. Add vertically and l zontally. Finally add the new column to the right the sums of the given columns. These two sums wil equal if your additions have been made correctly. W What is this number?

3.

5674	3895	4769	4906	5087	7459	????
8456	8407	9574	7549	7408	5764	????
6758	5744	3758	5754	4075	6578	????
5749	6738	7493	8038	8747	5643	????
7034	7408	6828	4753	3584	8539	????
4567	8573	9250	6489	6637	3285	????
????	????	????	????	????	????	????

4.

4536	7463	8467	8507	5076	8457	????
7595	5493	7409	6579	6504	9467	????
8508	6483	4506	7598	9587	6549	????
5987	7469	7608	6769	6846	7895	????
6479	6407	8957	4507	5849	7658	????
9684	6497	6746	9657	7593	6849	????
5875	7906	5864	5978	6859	8573	????
7690	6058	7605	6807	8593	7868	????
6457	8673	8675	8697	6548	8697	????
????	????	????	????	????	????	????

5. Add similarly each set of columns on page 17.

10. Excesses of 9's.—It is shown by literal numbers that a number divided by 9 gives the same remainder as the sum of its digits divided by 9. Try this with 4539. The remainder found by dividing a number by 9 is called the **excess of 9's** of the number. The process of finding excesses of 9's is also called **casting out the 9's.**

Note that only the remainder is considered. Hence any 9, or sums making 9, are omitted in adding the digits. Thus, in finding the excess of 25,347 drop $2 + 7$ and $5 + 4$. This leaves 3 for the excess.

In finding the excess of 7143 first find the sum of its digits, which is 15. In the place of dividing by 9 merely add 1 and 5. Both results will be 6.

Find the excess of each number in the column to the left of Ex. 4 above.

11. Use of Excesses.—Excesses are used in checking operations. These checks will be found very simple to apply after they have once been learned.

To check addition the excess of each number is first found and these excesses added. Then the excess of this small sum is found, and, if it is the same as the excess of the sum of the numbers, the addition is likely correct.

2546	excess 8	$8 + 5 + 7 + 2 = 22$, whose excess is 4.
1823	excess 5	
5704	excess 7	The excess of 13,945 is also 4. What is your
3872	excess 2	conclusion regarding the addition?
13945		

If one or more digits in the sum had been made too large and other digits enough smaller so that the excess would be the same as in the correct sum, any error would not be shown; 14,755 and 13,945 have the same excess. Such mistakes seldom occur.

Make up three columns with ten numbers in each, and each number containing at least four digits. Add and check by casting out the 9's.

12. Subtraction.—Subtraction is the reverse of addition. It is finding a number that added to some given number will produce another given number. This is the simplest form of subtraction. It is the one a clerk uses in giving you change. Suppose that you have made a purchase for 65 cents and have given the clerk a dollar in payment. How would the clerk give you your change?

	1 and 3 make 4. Write 3 under the 1.
45.74	5 and 2 make 7. Write 2 under the 5.
28.51	8 and 7 make 15. Write 7 under the 8.
17.23	1, carried from 15, and 2 make 3, and 1 make 4.
	Write 1 under 2.

13. Checking Subtraction.—The best check for subtraction uses excesses. Find the excess of the subtrahend and take this from the excess of the minuend. This equals the excess of the difference between the two numbers if the subtraction is correct. In case the excess of the subtrahend is larger than that of the minuend, add 9 to the excess of the minuend.

6432	excess 6	6 − 4 = 2 (difference of excesses).
2947	excess 4	Excess of 3,485 is also 2. What is your conclusion
3485	excess 2	as to the work?

Check the subtraction in Art. 12 by use of excesses.

EXERCISES

1. What must you add to 8 to get 13? to 6 to get 14? to 12 to get 20? to 9 to get 15? to 7 to get 11?

2. How will a clerk give you change from a dollar if your purchase has been 85 ¢? 65 ¢? 45 ¢? 28 ¢? 17 ¢?

3. How will a clerk give you change from $ 5 if your purchase has been $ 1.25? $ 2.35? $ 1.85? $ 3.45? $ 3.89?

Carry out the following subtractions and check:

4. 45.36 23.15	7. 29.08 10.25	10. 135.42 68.25	13. 5.607 3.425
5. 46.06 12.35	8. 52.13 25.31	11. 230.35 153.28	14. 13.024 8.107
6. 82.05 27.42	9. 73.50 26.42	12. 405.07 252.14	15. 16.162 12.247

14. A Number Contest.—The teacher selects captains so that the class will be divided into teams of from 8 to 10 pupils. How many captains will your class have? The captains then choose sides.

Each pupil sits in his seat with pencil and paper ready to take down the exercises—from 5 to 20—read by the teacher. At a signal by the teacher all begin to work. The pupil first through raises his hand and is scored 1, the next 2, and so on. If two or more pupils are through at the same time they score their papers the same. Work ceases at a signal by the teacher. All who have not completed their work score their papers five more than the score of the last pupil who completed the work. Papers are exchanged with pupils from another team. For each unanswered question 2 is added to the score and for each mistake 5 is added to the score.

Each captain now brings the papers from his team to the teacher, who reads all the scores of team A. These are taken down by all the pupils, added, and divided by the number on the team to get the average score. The averages of the other teams are found in the same manner. The team with the lowest average score wins.

New sides may be selected for the next contest or the same teams may compete for some time. The records of all contests are averaged and posted at the end of each contest.

Let the first contest be 5 to 10 exercises in addition; the second 10 to 20 exercises in subtraction.

The team called the Blues in a certain junior high school made the following scores in five contests: 23, 20, 18, 19, 17. Find their average score. The Greys' scores for the same contests were 22, 23, 20, 19, 15. Compare their averages.

15. Problems.—Problems depending upon subtraction sometimes appear difficult. To solve such a problem reverse, or **undo,** an addition that has been carried out. It may make the solution clearer to show first the addition and then undo it. Use this principle often.

EXERCISES

1. The larger of two numbers is 375 and their difference is 164. What is the smaller number? $164 + ? = 375$.

2. The difference between two numbers is 425. If one number is 153, what is the other number?

3. The difference between two numbers is 352. If one number is 837, what **may** the other number be?

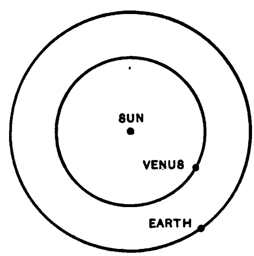
4. William's father made purchases at a store and gave the clerk a $ 10 bill in payment. What change should the clerk give him if the purchases were $ 2.35? if $ 1.47? if $ 3.29?

5. How would you give change, if you were a clerk and were handed $ 5 in payment for a purchase of $ 2.63? a purchase of $ 3.47? a purchase of $ 1.13?

6. An automobile cyclometer read 1,647.6 mi. at the beginning and 2,152.3 mi. at the end of a trip. How long a trip was taken?

7. Make up problems like the last three for the class to solve.

8. The planet Venus is 67200000 mi. and the earth 93000000 mi. from the sun. Divide these numbers into periods and read them. Make a drawing to show when the earth is nearest to Venus; when farthest away. Find each of these distances.



Hold a number contest, using 5 exercises each in addition and subtraction of decimals.

i. **Multiplication.**—Use the common form for long
:iplication previously learned.

'. **Order of Multiplication.**—In how many different or-
can you find $5 \times 7 \times 6$? What are these products?
ch could you find the more easily? Always be on the
out for the simplest way to carry out multiplications.

}. **Checking Multiplications.**—Find the excess of the
:iplicand and of the multiplier. Next find the product
ıese two numbers and find its excess. If this equals the
ss of the product of the two numbers, the work is likely
ect. Thus, $256 \times 32 = 8192$. The excess of 256 is 4
of 32 is 5; 4×5 is 20 which has an excess of 2. The
ss of 8192 is also 2. Hence, what is your conclusion?

EXERCISES

Give the products of the digits in Ex. 1, page 5, in the
e of the sums called for there.

How do we multiply by 10? by 100? by 1000?

Multiply each number in Exs. 7–16, page 4 by 10;
.00; by 1000.

arry out the following operations and check all requir-
pencil and paper:

ŀ. $7 \times 5 \times 6$

i. $5 \times 73 \times 2$

i. $3 \times 25 \times 12$

r. $4 \times 67 \times 50$

}. $25 \times 42 \times 6$

). $460 \times 2 \times 5$

). $48 \times 27 \times 5$

L. $53 \times 4 \times 25$

!. 645×37

}. 675×84

14. $8{,}075 \times 56$

15. $7{,}056 \times 69$

16. $85{,}067 \times 48$

17. $45 \times 6 \times 7$

18. $4{,}758 \times 49$

19. $45 \times 17 \times 2$

20. $250 \times 713 \times 4$

21. $32{,}450 \times 576$

22. $2 \times 46 \times 35$

23. $40{,}367 \times 305$

19. Division.—Division is the reverse of multiplication. To divide 312 by 13 means to find the number which, multiplied by 13, gives 312. Use the common form of short division with small divisors and the common form of long division with large divisors. Note that, if the second digit of the divisor is small, as 72 and 6,382, the first digit can be used to find each digit of the quotient. If the second digit in the divisor is large, as 78 and 3,705, the first digit in the divisor **increased by one** can be used to find each digit in the quotient.

20. Checking Division.—Find the excess of the divisor and of the quotient. Multiply these excesses together and find the excess of their product. Add to this the excess of the remainder and find the excess of this result. If this last excess found equals the excess of the dividend, the work is likely correct. Thus, $25{,}123 \div 47 = 534$ and remainder 25. Excess of 47 is 2 and of 534 is 3. $2 \times 3 = 6$ which is also its excess. The excess of 25 is 7, and $6 + 7 = 13$, which has an excess of 4. As the excess of 25,123 is also 4 the division is likely correct.

EXERCISES

1. In Exs. 2, 3, and 4, page 5, give the quotient of the upper number divided by the lower number.

2. How do you divide by 10? by 100? by 1000?

3. Divide the numbers in Ex. 1, page 4, by 10; by 100.

Carry out the following operations and check:

4. $7{,}328 \div 713$	7. $32{,}045 \div 258$	10. $56{,}003 \div 507$
5. $4{,}073 \div 408$	8. $40{,}308 \div 509$	11. $40{,}307 \div 409$
6. $3{,}704 \div 456$	9. $30{,}605 \div 357$	12. $28{,}034 \div 435$

21. Problems.—A given number in a problem has often been obtained by multiplying together a given and an unknown number. The problem is then solved by **undoing** the multiplication.

EXERCISES

1. What number multiplied by 4.28 gives 74.137 as a product? **Undo** the multiplication. How is multiplication undone?

$$\begin{array}{r} ????? \\ \underline{4.28} \\ 74.137 \end{array}$$

2. How far will a train go in 7.3 hr., which averages 29.25 mi. per hour?

3. A train went 237 mi. in 8.25 hr. How many miles did it average per hour?

4. William bought a football from James for which he paid $ 0.75. This was only .6 of what James paid for the football. How much did James pay?

5. Mamie raised 17 chickens during the summer, which she sold in the fall. If the total weight was 59.5 lb., what would be the average weight?

6. Mr. Williams paid $ 7.50 per ton for coal, which was an increase of .25 of the price last year. How much did he pay per ton last year?

7. A football team of 15 players made a trip which cost $ 25.05 in all. What were the expenses per player?

8. Each of the 139 pupils of a junior high school pledged themselves to sell 12 thrift stamps at 25 cents each. How much would all the thrift stamps be worth?

9. The 168 pupils in a certain junior high school sold thrift stamps for $ 347.75. How much was this per pupil? For how much should a school of 347 pupils sell thrift stamps at the same rate?

22. Powers.—In the place of 3×3 we write 3^2; in the place of $5 \times 5 \times 5$ we write 5^3, etc. We say that 3 is raised to the **second power**, 5 to the **third power**, and so on.

23. Signs of Operation.—The order in which several operations are to be carried out is often a confusing thing. Mathematicians have agreed to carry out the operations in the following order:

1. *Raise to powers and extract roots;*
2. *Multiply as indicated by "of";*
3. *Multiply and divide in the order in which these operations arise;*
4. *Add and subtract in any order.*

$$\begin{aligned} &49 + 12 \div 3 \times 2 + 17 \times \tfrac{2}{3} \text{ of } 6 - 6^3 \div 9 \times \sqrt{16} \\ &\quad = 49 + 12 \div 3 \times 2 + 17 \times \tfrac{2}{3} \text{ of } 6 - 216 \div 9 \times 4 \\ &\quad = 49 + 12 \div 3 \times 2 + 17 \times 4 - 216 \div 9 \times 4 \\ &\quad = 49 + 4 \times 2 + 68 - 24 \times 4 \\ &\quad = 49 + 8 + 68 - 96 \\ &\quad = 29. \end{aligned}$$

EXERCISES

Carry out the following operations:

1. $23 + 5 \times 6 - 12 \div 4$
2. $19 - 60 \div 2^2 + 5 \times 3^2$
3. $3 \times 12 \div 4 - 2^2 \times 3$
4. $9^2 + 3 \times 5 - 4 \div 2^2$
5. $54 - \frac{2}{3} \text{ of } 6 + 16 \div 2^3$
6. $25 - \frac{3}{4} \text{ of } 12 + 5^2 \times 2$
7. $12 + 2 \times 7^2 - 10 \times 5 \div 2$
8. $47 - 50 \div 5^2 - 5 \times 3^2$
9. $2 + \sqrt{36} \div 3 - \frac{1}{2} \text{ of } 14 \div 7$
10. $33 + 250 \div 5^3 - 3 \times 4^2$
11. $9 + \sqrt{25} \times 7 - 2^2 \times 3 + 2$
12. $17 - 2^2 \times \sqrt{49} + \frac{3}{11} \text{ of } 33 - 3$

24. Useless Decimal Places.—Decimal places which have no meaning often arise in computations. In paying $ 24.5732 the third and fourth decimal places have no meaning. If the first decimal place of those having no meaning is 5 or more, the first usable decimal place is increased by 1. Thus, $ 7.3562 would be $ 7.36 while $ 19.3429 would be only $ 19.34. State what each of the following would be if only two decimal places were usable: 26.3471; 8.0642; 605.325; 850.2345; 30.5208.

Results which are to contain two decimal places are usually spoken of as being correct to .01; three places as correct to .001, etc. How many decimal places would the final result contain which you are to carry out to .1? to .0001? to .01? to .00001?

25. Abbreviated Addition of Decimals.—In finding the sum of the accompanying column correctly to .01, we disregard the last column altogether. The third column adds up to 19, or to .019. That is, 9 would be placed in the third decimal place and 1 carried to the second decimal place. By what we have just seen above, the 9 in the third decimal place would increase the second decimal place by 1. Hence we carry 2 instead of 1 to the column of the second decimal place.

21 22
35.1641
7.2385
127.5263
19.4813
189.41

EXERCISES

1. What would you pay for $\frac{1}{2}$ doz. oranges that sell at 60 cents per dozen? at 65 cents per dozen?

2. William has averaged one misspelled word in every 10 words. At this rate, how many words would he misspell out of 80 words? of 73 words? of 97 words?

3. In the first column of Ex. 4, page 17, read the numbers correctly to .01. Also read these numbers correctly to .1.

Make the vertical and the horizontal additions on the opposite page correctly to .01. Is the work incorrect if the

sum of the sums in the vertical columns does not equal the sum of the sums in the horizontal column to the right? How much of a difference would be permitted before calling the work incorrect?

4.

4.	45.634	135.04523	305.3406	??????
	178.7562	86.346	427.563	??????
	63.058	35.805	37.7594	??????
	379.4953	365.468	385.48	??????
	43.627	92.056	172.735	??????
	264.045	339.4352	7.472	??????
	54.735	53.0724	306.528	??????
	7.05	29.4563	38.6073	??????
	405.6732	7.7438	352.453	??????
	74.584	450.576	48.35	??????
	653.7462	47.0307	63.5326	??????
	46.5305	350.841	5.6472	??????
	??? ??	??? ??	??? ??	

5.	753.4203	362.0642	107.8423	??????
	94.6307	159.4631	64.5398	??????
	237.7456	28.795	573.6348	??????
	19.5349	306.1364	39.4517	??????
	450.796	63.498	723.8546	??????
	38.5403	195.7346	47.3728	??????
	139.6437	28.0645	389.5647	??????
	634.45	7.341	36.061	??????
	30.637	64.056	19.75	??????
	194.3527	17.67	203.875	??????
	74.453	407.542	74.5683	??????
	564.8603	91.572	37.4735	??????
	??? ??	??? ??	??? ??	

Hold a number contest on abbreviated addition of decimals.

26. Abbreviated Multiplication.—Multiplication of decimals is also abbreviated, as in the accompanying illustration of finding the product of 35.21 × 24.36 to .01.

24.36	
35.21	
121.80	(1)
730.8	(2)
4.87	(3)
.24	(4)
857.71	(5)

1. Multiply 24.36 by 5—two decimal places.
2. Multiply 24.36 by 30—one decimal place.
3. Multiply 24.36 by .2—three decimal places but only two used. Run a line through 6 in the multiplicand, showing that it need not be considered after this.
4. Multiply 24.3 by .01, giving .243 of which only .24 is used.
5. Add the partial products.

EXERCISES

Carry out the following, as in Art. 26, to .001:

1. 24.15 × 3.14
2. 16.36 × 6.45
3. 53.16 × 4.6
4. 42.35 × 6.75
5. 60.73 × 4.16
6. 36.24 × 4.25
7. 34.51 × 5.12
8. 26.18 × 3.15
9. 8.305 × 6.8
10. 75.208 × 4.02
11. 30.5 × 0.16
12. 28.15 × 2.34
13. 63.5 × 6.12
14. 204.63 × 3.7
15. 47.045 × 5.08
16. 32.45 × 4.18

Carry out the following, as in Art. 26, to .01:

17. 9.07 × 0.156
18. 24.052 × 0.106
19. 6.043 × 0.85
20. 36.4503 × 6.7
21. 64.96 × 4.13
22. 7.483 × 2.07
23. 31.067 × 6.17
24. 6.065 × 8.05
25. 13.405 × 3.07
26. 50.64 × 5.18
27. 48.19 × 0.06
28. 19.06 × 6.08
29. 45.036 × 0.045
30. 14.305 × 2.05
31. 25.06 × 9.07
32. 36.07 × 8.09

Carry out a number contest, using abbreviated multiplication of decimals.

27. Abbreviated Division.—If the divisor is a decimal, first multiply both divisor and dividend by the smallest power of 10 which will make the divisor a whole number. Then proceed as in the illustration.

1.496	
$4376\overline{)6545}$	Divide by 4376.
$\underline{4376}$	
2169	Cut off the 6 in the divisor and divide by 437.
$\underline{1750}$	4×6 gives 2 to carry to 4×7.
419	Cut off the 7 in the divisor and divide by 43.
$\underline{393}$	9×7 gives 6 to carry to 9×3.
26	Cut off the 3 in the divisor and divide by 4.
$\underline{26}$	6×3 gives 2 to carry to 6×4.

EXERCISES

1. Multiply 3.4653, 4.0083, 0.0036027, 0.01050407, 30.4056, and 2.0040507 by 10; by 100; by 1000.

2. By what must each of the following be multiplied so that the product will be a whole number: 3.045? 0.00354? 4.5006? 0.34002?

Carry out the following, as in Art. 27, to .01 and check:

3. $3.456 \div 261$
4. $16.032 \div 2.53$
5. $4.307 \div 5.61$
6. $20.527 \div 6.17$
7. $35.004 \div 3.52$
8. $27.105 \div 0.233$
9. $14.607 \div 7.88$
10. $7.428 \div 3.02$
11. $46.004 \div 7.61$
12. $3.305 \div 3537$
13. $0.3064 \div 2.345$
14. $1.345 \div 503$

Carry out the following, as in Art. 27, to .001 and check:

15. $4.3645 \div 236$
16. $0.3562 \div 3.87$
17. $5.3071 \div 2.118$
18. $2.8305 \div 217$
19. $17.0453 \div 2.135$
20. $6.1109 \div 23.15$
21. $34.064 \div 385$
22. $71.3062 \div 216$
23. $32.7105 \div 35.8$
24. $63.009 \div 20.04$

Carry out a number contest, using abbreviated division of decimals.

28. Approximations.—Often the exact result is not needed but only an approximation of the result. For instance, we may wish to know **about** what a certain trip will cost; **about** how far it is to a certain city; or **about** how long a time it will take to do a certain piece of work.

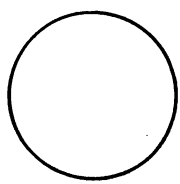
In solving a problem, or in making a computation, it is a good plan first to approximate the result and later to compare this with the computed result. For instance, the product of 24.35 times 31.56 is nearly 25 times 32. We know that 25 × 32 is $\frac{1}{4}$ of 3,200 or 800. How? If we then find that the computed result is around 80, we are quite sure that an error has been made in the placing of the decimal point.

Remember that addition and subtraction are reverse processes, as are also multiplication and division.

EXERCISES

In solving the following problems, first estimate the result and then compare the computed result with the estimated one. Carry results to .01 of first ten exercises.

1. Find the cost of 17.6 yd. at $ 1.35 per yard.



2. The circumference of a circle is found by multiplying the diameter by 3.1416. Find the circumference of the circle whose diameter is 3.15 ft.; 2.36 yd.; 18.25 in.

3. Find the diameter of the circle whose circumference is 134.52 in.; 215.67 in.; 45.36 yd.; 28.32 ft.

4. The area of a circle is found by squaring its radius and multiplying the result by 3.1416. Find the area of the circle whose radius is 2 ft.; 0.45 rd.; 4.53 yd.; 0.56 ft.

5. The product of two numbers is 36.48. If one number is 2.4, what is the other number?

6. The product of two numbers is 0.194. If one number is 3.14, find the other number.

7. What number multiplied by 19.46 gives 7.185?

8. What number divided by 34.15 gives 63.28?

9. By what must 0.06732 be divided to give 23.07?

10. What number divided by 0.19 gives 14.06?

11. What is the cost of 5.46 T. of coal at $ 5.65 per ton? To how many decimal places should the computation be carried?

12. How many acres are there in a piece of land 86.5 rd. by 40.25 rd.?

13. If the land in the last problem is to be sold for $ 100 per acre, to how many decimal places should the land be found in acres? Carry this out.

14. Mr. Clark bought 8,460 lb. of coal. How many tons is this? See how simply you can carry out the computation.

15. A meter is 39.37 in. long. Find the length and width in meters of a rug 9 ft. by 12 ft.

16. A grazier bought 132 calves for $ 4,900. About what was this per head?

A lady looked at some cloth which was priced at $ 1.65 per yard. She also looked at a remnant of 7 yd. of the same quality, which was priced at $ 9.75.

17. About what is the price per yard of the remnant?

18. About what is the price of 7 yd. of the first piece of cloth?

19. About how much did the lady save if she bought the remnant?

20. Find the length and width of your schoolroom. About what is the area of the floor in square feet? in square yards?

29. Metric System.—In Book I we have already learned how the simple metric system was invented and adopted in France. It is a decimal system based upon 10, like the simple Hindu number system and the United States money system. Turn to the back of the book and compare the metric tables of weights and measures with the English tables. Note the close similarity of the first part of the metric tables with the United States money table. In reviewing the metric tables first repeat the United States money table.

Notice in the tables that we write 9 m.^{2} in the place of 9 sq. m., and 13 cm.^{3} in the place of 13 cu. cm. See Arts. 22 and 40. Another abbreviation for cubic centimeters is c.c.; thus, 13 c.c.

30. Reductions.—In reducing 6 Km. 7 Dm. 9 m. 5 cm. to any denomination first change it to Kilometers and a decimal part of a Kilometer. Do this without taking time to multiply by the 10's. As there are no Hektometers nor decimeters these places must be filled by zeros. Thus we have 6.07905 Km. This is reduced to any other denomination by merely changing the decimal point. Thus,

6.07905 Km. = 6079.05 m.

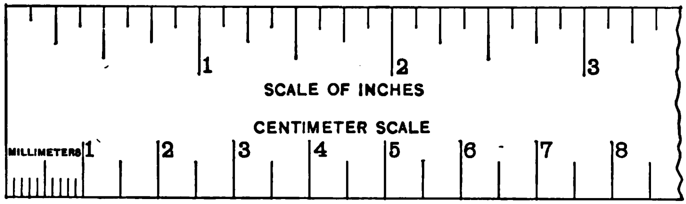
Change to Dekameters; to centimeters; to Hektometers; to decimeters.

Other reductions are carried out similarly. Thus,

5 Kg. 2 Hg. 9 g. 3 dg. is reduced to grams by writing from the left all the parts in order up to grams, then a decimal point, followed by the decimal part. Thus,

5 Kg. 2 Hg. 9 g. 3 dg. = 5209.3 g.

Change to decigrams; to Hektograms; to Kilograms; to Dekagrams.



EXERCISES

1. Measure the length and width of your desk in metric units. Express each in centimeters; in decimeters; in meters.

2. Measure the length and the width of your schoolroom and express each in meters and decimeters. Change each to meters and a fraction of a meter. Change each to decimeters.

3. Suggest some other lengths in your schoolroom to measure. Before making the measurements estimate the length. Did you make a large error? Try to make a smaller error at the next estimate. Express each of these distances in meters; in decimeters; in centimeters.

4. Change 5 Kg. 7 Hg. 3 g. 2 dg. to Kg.; to g.; to Dg.

5. Change 1547.32 g. to Dg.; to Kg.; to dg.

6. Change 6 Km. 3 Hm. 7 Dm. 7 dm. to Dm.; to m.; to cm.

7. Change 3 Hl. 5 l. 3 dl. to l.; to dl.; to Kl.

8. Change 4 Km. 3 Dm. 7 dm. to m.; to Hm.; to Km.

9. 1 sq. Dm. equals how many square meters? Hence, 314 sq. m. equals how many square Dekameters?

10. 71 Dm.^2 equals how many square meters? how many square Hektometers? how many square decimeters?

11. Make up two problems like Exs. 9 and 10 for the class to solve.

31. Addition, Subtraction, Multiplication, Division.— These operations are carried out first by reducing all the units to one denomination. The operations are then carried out just as with any decimal. Thus, to multiply

5 Hm. 7 Dm. 8 dm. by 6 change it to 6 times 5.708 Hm., which gives 34.248 Hm.

To find how many 3 dl. 5 cl. there are in 8 l. 4 dl., each is first reduced to one denomination as 35 cl. and 840 cl. 840 is then divided by 35, which gives 24.

EXERCISES

1. Add 6 Kg. 7 Hg. 9 g.; 3 Hg. 7 Dg. 9 g.; 4 Kg. 3 Dg. 7 g.; 3 Kg. 7 Hg. 6 Dg. 7 g.

2. From 8 Dl. 6 l. 5 cl. take 6 Dl. 7 l. 4 dl.

3. Multiply 8 Km. 6 Hm. 7 m. 3 dm. by 6; by 57; by 576.4.

4. Divide 675.34 g. by 4.6; by 45.37; by 6.73.

5. Divide 8 Dm. 4 m. 3 dm. into parts 1.4 m. each; 2.3 m. each.

6. What is the cost of 2 Dm. 6 m. 7 dm. of cloth at $ 2.35 per meter?

7. Milk sells at $ 0.125 per liter. What is the cost of 3 Dl. 8 l. 5 dl.? of 6 Hl. 9 l. 5 dl.?

8. An automobile goes 263 Km. 8 Hm. in $7\frac{1}{2}$ hr. What was the average speed of the automobile per hour?

9. How many metric tons are there in 36452 Kg.? in 30426 Kg.? in 485 Kg.?

10. At $ 6.50 per metric ton what is the cost of 8500 Kg. of coal? of 15400 Kg.? of 450 Kg.?

11. Make up for the class to solve, a problem on buying by the meter; buying by the liter; buying by the Kilogram; buying by the metric ton.

32. Areas and Volumes.—First become familiar with the tables of square and cubic measure found on pages 225 and 226. Note their simplicity compared with the corresponding English tables. To find the area of a rectangle 4 m. 5 dm. by 3 m. 7 dm. change both to decimeters, that is, to 45 dm. by 37 dm. The area is hence,

$$45 \times 37 \text{ sq. dm.} = ?$$

This is how many square meters?

EXERCISES

1. Find the area of a rectangle 5 dm. 3 cm. by 2 dm. 6 cm. in both square decimeters and square centimeters.

2. James has a garden 3 Dm. 7 m. by 2 Dm. 8 m. Find its area in square Dekameters; in square Hektometers.

3. A square Hektometer, called a hektare, is used in measuring land. James's garden is how many hektares?

4. The radius of a cylindrical tank is 0.4 m. and the depth is 0.8 m. Find its volume in cubic meters; in cubic decimeters. How many liters does the tank hold?

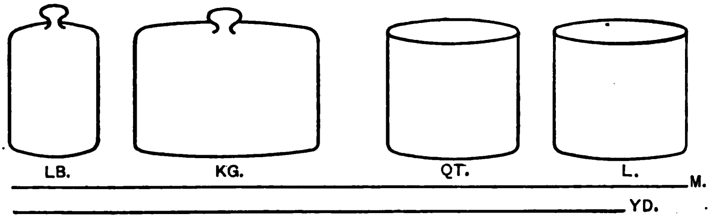
5. What is the weight of a cubic centimeter of water? What is the weight of a cubic decimeter of water? What is the weight of a liter of water?

6. A tank in a junior high school laboratory measures 6 dm. 8 cm. by 4 dm. 7 cm. by 3 dm. 9 cm. Find its volume in cubic decimeters; in cubic centimeters; in liters.

7. Find the volumes of any tanks in your laboratory.

8. Find the volume of your schoolroom in cubic meters.

9. Find the area of the top of your desk in square decimeters.



33. Relation Between English and Metric Units.—Note carefully the above comparisons, as well as the following numerical relations:

1 m. = 3.3 ft., *nearly.*
1 Km. = .6 mi., *nearly.*
1 Kg. = 2.2 lb., *nearly.*
1 l. = 1 qt., *nearly.*
1 metric ton = 1.1 T., *nearly.*

In solving the following exercises make use of these equivalents. State your estimates, as far as possible, without use of pencil and paper, as whole numbers or halves, but no smaller fractions.

EXERCISES

1. In buying 10 lb. sugar you would ask for how many Kilograms?

2. How many meters would you ask for in buying 10 yd.? 6 yd.? 15 yd.? 100 yd.?

3. How many metric tons would you buy if buying 4 T.? 12 T.? 20 T.? 15 T.?

4. In buying flour how many Kilograms would you ask for in order to get about 10 lb.? 24 lb.? 48 lb.? 96 lb.?

5. Gene had a letter from his brother Wendell in France saying that his brother had been on a march of 60 Km. About how many miles was this?

6. The Eiffel Tower is 300 m. high. About how many yards high is it? About how many feet high is it?

The children of a class in a certain junior high school are sending semiyearly $ 18.25 to little Berthe, one of the fatherless children in France.

7. The value of each franc is 20 cents. How many francs will these children send Berthe each year?

8. There are 73 pupils in this class. How much does each contribute yearly to this fund?

9. They sent her $ 5.00 as a Christmas gift. How many francs was this?

10. There are 100 centimes in a franc. Berthe bought 3 m. of cloth for a dress with part of this money. She paid 2 francs 40 centimes per meter for the cloth. How much did she pay for the cloth? How much was this in United States money?

The class received a letter from Berthe in which she said that she was 11 yr. old, weighed 28.6 Kg., and was 1.25 m. tall.

11. About what was Berthe's weight in pounds?

12. About what was her height in feet and inches?

13. Berthe also wrote that she lived 375 Km. from Bordeaux and 220 Km. from Paris. About how many miles does she live from Bordeaux? from Paris?

14. The average weight of the pupils of this class was 64 lb. How many Kilograms would they call this in writing to little Berthe?

15. Their average height was nearly $4\frac{1}{2}$ ft. What would this be in metric units?

Hold a number contest, using metric measures.

II

LITERAL NUMBERS

34. Mathematical Shorthand.—Statements can be written very simply and so that they can be used in computations by the use of a mathematical shorthand. Letters and symbols are used in the place of words. For instance, the circumference of a circle equals two times π, $\left(\frac{22}{7}\right)$, times its radius, is written

$$C = 2 \times \pi \times R.$$

EXERCISES

Give the following in mathematical shorthand:

1. Minuend minus subtrahend equals difference.
2. Minuend equals subtrahend plus difference.
3. Minuend minus difference equals subtrahend.
4. Minuend minus subtrahend minus difference equals zero.
5. Dividend equals divisor times quotient plus remainder. Use D and d.
6. Dividend minus the remainder equals the divisor times the quotient.
7. Total sales equal purchase price plus profits, or purchase price less losses.
8. Money on hand at the beginning of the day's business plus sales equals money on hand at the end of the day's business plus expenditures.
9. The circumference of a circle divided by its diameter equals π.

35. Numerical and Literal Expressions.—A statement in only numerals and symbols is called a **numerical expression,** as $5 \times 4 + 7$. A statement also containing letters is called a **literal expression,** as $3 \times a + 6 \times b + 9$.

36. Formulas.—Mathematical rules and laws can be stated very simply by literal expressions. Such literal statements are called **formulas.**

The curved surface of a cone equals π times the radius of its base times its slant height. Or,

$$S = \pi \times R \times sh. \qquad (1)$$

The number of gallons in a rectangular tank is the product of the dimensions in feet times 7.48. Or,

$$G = L \times W \times H \times 7.48. \qquad (2)$$

EXERCISES

Which are numerical and which literal expressions?

1. $2 + 5$	4. $a - b$	7. $2 \times a + b - c$
2. $8 \times a + g$	5. $14 - 9$	8. $2 \times 5 + 3 - 4$
3. $3 \times a - v$	6. $9 \div 2m$	9. $2 + 3 - r$

10. The number of bushels of oats a bin will hold equals length times width times height, all in feet, divided by 1.25. Express this by a formula.

11. To find how many tons of hard coal a rectangular bin will hold, multiply the length, width, and height together in feet and divide by 28. Express by a formula.

12. The price of any number of articles equals the price of one times the number of articles. Use P and p for the two prices and express this as a formula.

37. Evaluating Literal Expressions.—Literal expressions are **evaluated** by replacing the letters by their numerical values. Thus, to find the number of gallons held by a rectangular tank 9 ft. by 4 ft. by 1.5 ft., replace the corresponding letters in formula (2) on page 29. Hence,

$$G = L \times W \times H \times 7.48 = 9 \times 4 \times 1.5 \times 7.48 = ?$$

This process is also called **substitution.**

EXERCISES

Find the value of each of the following literal expressions when $m = 16$, $n = 4$, $a = 2$, $b = 3$, $c = 1$:

1.	$m - a$	**5.**	$3 \times m + n$	**9.**	$5 \times a \times a + b \times b$
2.	$m \div n$	**6.**	$9 \div b + c$	**10.**	$b - c \times a + n$
3.	$m + a$	**7.**	$m - n + n$	**11.**	$m \div n + a \times b$
4.	$2 \times m$	**8.**	$27 \div b + c$	**12.**	$b + b + b + b$

13. Rewrite the formula for Ex. 8, page 28. Were there any errors in a store where for one day cash on hand in the morning was \$ 56.17 and at night \$ 124.12, sales \$ 97.48, expenditures \$ 29.53 ? cash on hand in the morning \$ 35.67 and at night \$ 65.68, sales \$ 49.28, expenditures \$ 18.27 ?

14. Express as a formula that the sum of two numbers less the smaller number equals the larger number. Find the larger number if the sum is 314 and the smaller number is 98; if the sum is 103 and the smaller number is 47.

15. Electricians use the formula, $V = A \times R$. Find the numerical value of V when $A = 15$, and $R = 47$; when $A = 8.6$, and $R = 28.7$.

Hold a number contest on writing formulas.

38. Multiplication Signs.—In literal numbers multiplication signs are replaced by a dot or omitted. Thus, $2 \times \pi \times R$ is $2 \cdot \pi \cdot R$, or more simply $2 \pi R$.

39. Coefficient.—The numerical part of a literal expression is called its **coefficient**. Such are 5 in $5mn$, and 9 in $9WGH$. If no coefficient appears, 1 is understood, as ab.

40. Exponent.—In the place of writing $a \times a$ or $w \times w \times w \times w$ we write a^2 and w^4. These small numbers above to the right of a quantity are called **exponents** or **indices.** We read q^2 as q square, m^3 as m cube, k^4 as k raised to the fourth power, k to the fourth degree, or k with an index 4, or k fourth. When no exponent appears, 1 is understood, as $5abc$.

In the metric system of measures 5 square meters is written 5m^2; 7 cubic centimeters is written 7cm^3. See Art. 29.

EXERCISES

1. Name the coefficients and the exponents in: $5m^3$; $7n^2h$; $9m^3n$; $47q^5r^3$; $9g^4h^7$; $43m^3v^6$; $19t^3r^2$; $7g^4h^8$; $916x^3y^2$. Read these literal numbers.

Write the following in the simplest form by the use of coefficients and exponents:

2. $r \times r \times r$.
3. $6 \times m$.
4. $9 \times j \times k$.
5. $r \times s \times 5$.
6. $g \times g \times 7$.
7. $8 \times t \times u \times u$.
8. Five m cube.
9. Seven times the fourth power of f.
10. Eight times k square.
11. q times the fourth power of w.
12. m square times n cube.
13. Three times x square times y cube.
14. Nine times a number equals sixty-three.
15. Seven times the fourth power of a number equals five hundred seventeen.

16. Eighteen square centimeters; 25 cubic meters.

17. Thirty-seven and five-tenths cubic decimeters; 9 square Kilometers.

The space, s, in feet, that a body falls from rest in any number of seconds, t, is given by the formula,

$$s = 16t^2.$$

The number of feet that a stone would fall in 3 seconds is, then,

$$\begin{aligned} s &= 16 \times 3^2, \\ &= 16 \times 9, \\ &= 144. \end{aligned}$$

18. From the above formula find how far a baseball will fall in 1 sec.; in 5 sec.; in 10 sec.

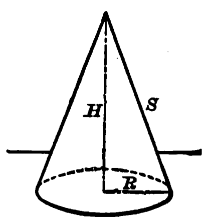
19. The velocity in feet per second that a body will gain in falling any number of seconds is 32 times the number of seconds. Express this as a formula in which t is used for the time in seconds.

20. A falling stone will gain a velocity of how many feet per second in falling 5 sec.? 10 sec.? 20 sec.?

21. A body falling from rest during any one particular second will fall a distance which is 16 less than 32 times the number of the second. Express this as a formula.

22. How far would a stone fall during the third second? during the tenth second?

23. The total area of the surface of a cone equals π times the square of the radius of its base plus π times the radius times the slant height. Explain the following formula of this statement,



$$A = \pi R^2 + \pi RS.$$

Hold a number contest using exercises on literal numbers.

24. Find the total area of the surface of a cone whose slant height is 6 ft. and the radius of whose base is 3 ft. What will be the unit of measure of the area?

25. Find the total area of the surface of a cone whose slant height is 15 dm. and the radius of whose base is 9 cm. What will the unit of measure be?

26. Turn to page 226 and explain each of the formulas.

Write the formula needed for each of the following exercises; then evaluate the formula for the numbers given:

27. Surface of a sphere whose radius is 3 in.; 32 cm.

28. Area of a circle whose radius is 3 in.; 5 cm.

29. Volume of a cone whose altitude is 12 in. and the radius of its base 3 in.

30. Volume of a cone whose altitude is 15 cm. and the radius of its base 4 cm.

31. Volume of a pyramid whose altitude is 8 in. and its base a square whose area is 6 sq. in.

32. Volume of a pyramid whose altitude is 1.6 dm. and area of its base 2.4 sq. dm.

33. The entire surface of the cylinder whose altitude is 9 in. and the radius of its base 3 in.

34. The entire surface of a cone whose slant height is 2.4 dm. and the radius of its base .5 dm.

35. State by a formula that the perimeter of a rectangle equals twice its length plus twice its width.

36. Find the perimeter of a rectangle 11 ft. by 9 ft.

37. The weight of a load of coal (called *gross*), less the weight of the empty wagon (called *tare*), equals the weight of the coal (called *net*). State this by a formula.

38. A wagon full of coal weighs 5670 lb. and the empty wagon 1940 lb. Find the weight of the coal.

39. State a mathematical law for the class to express as a formula.

41. Equations.—All expressions containing the equality sign, =, indicate that two numbers have the same value. Such expressions are called **equations.** Many illustrations have arisen in the last few lessons. What are some of them? The parts on each side of the equality sign are called the left and the right hand **members** of the equation.

42. Solution of Equations.—The equation $7n = 35$ indicates that 7 times a number, n, equals 35. For what value of n is this true?

In finding the value of a letter in an equation, as was done above, we are said to have **solved the equation.** The numerical value of the letter is called the **root** of the equation. What is the root of the equation,

$$4n = 12?$$

EXERCISES

As far as possible solve the following without the use of pencil and paper:

1.	$6a = 18$	**8.**	$8a = 4$	**15.**	$13m = 2.47$
2.	$5k = 35$	**9.**	$.5T = .2$	**16.**	$192a = 21.12$
3.	$15z = 75$	**10.**	$4.8T = .96$	**17.**	$3.6c = 5.40$
4.	$7z = 21$	**11.**	$4T = 3$	**18.**	$32b = 3840$
5.	$2.5k = .75$	**12.**	$88r = 880$	**19.**	$44a = 4444$
6.	$4p = .2$	**13.**	$50q = 3.5$	**20.**	$116R = 2320$
7.	$10p = 25$	**14.**	$36T = 0$	**21.**	$19.8m = 5940$

The literal number may be either in the left or in the right hand member of the equation. Thus,

$$12 = 3n, \ ? = n.$$

Similarly solve:

22.	$16 = 4b$	**26.**	$13 = 1.3a$	**30.**	$5.1 = 17a$
23.	$32 = 2a$	**27.**	$18 = 36a$	**31.**	$0 = 13a$
24.	$39 = 3T$	**28.**	$188 = 2n$	**32.**	$400 = 16g$
25.	$3.5 = .7c$	**29.**	$12.5 = 25T$	**33.**	$5.36 = 14W$

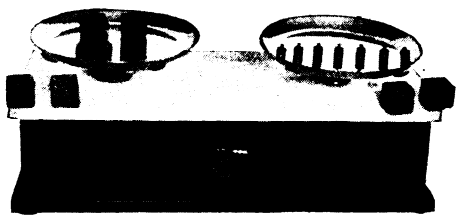
Hold a number contest on writing and solving equations.

43. Further Equations.—The equation says that 4 times a number, n, added to 3 is as large as 2 times the number added to 7. The balance scale also says the same thing.

$$4n + 3 = 2n + 7$$

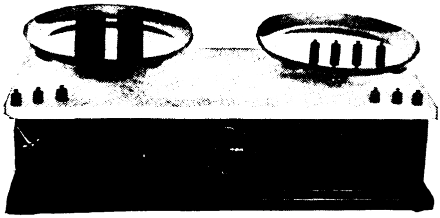
If the same number of blocks be added to each pan, the scales will still balance. If the same number of blocks be taken from each pan, it will also balance. All the blocks are taken from one pan and the same number from the other pan. This leaves the pans as in the picture to the right. The number of blocks and weights in each pan is shown by the equation above the scales.

$$2n + 3 = 7$$



If the known weights in the pan with the blocks be removed and a like number from the other pan, the balance is still preserved. One pan now contains only blocks and the other pan only known weights as here shown.

$$2n = 4$$



The corresponding equation is like the equations solved on the last page. How many known weights does each block weigh?

If a number is subtracted from one member of an equation, that number must first be added to both members. Thus,

$$5a - 4 = 2a + 11, \qquad (1)$$
$$5a - 4 + 4 = 2a + 11 + 4, \qquad (2) \text{ adding 4 to both members.}$$
$$5a = 2a + 15. \qquad (3) \text{ How?}$$

Find the value of a.

44. Principles Used in Solving Equations.—The following operations which leave true equations are used in solving equations:

1. *Adding the same number to both members.*
2. *Subtracting the same number from both members.*
3. *Multiplying both members by the same number.*
4. *Dividing both members by the same number.*

45. Checking Solutions.—The solution of an equation is not really completed until it has been checked. This is done by substituting the root in the **very first** equation to see if this leaves a true equation. Try substituting 2, 5, and 3 for a in (1) above. What is your conclusion?

EXERCISES

Solve the following equations and check:

1. $3t + 2 = 14$
2. $5r + 7 = 19$
3. $45 = 9 + 6g$
4. $8 + 2a = 36$
5. $13b + 9 = 35$
6. $51 = 7c + 2$
7. $15 + 4g = 75$
8. $a - 1 = 7$
9. $2t - 3 = 6$
10. $.7a - 5 = 9$
11. $36 = 12 + k$
12. $40y - 7 = 25$
13. $1.7s - 2 = 32$
14. $14 + 9g = 122$
15. $2.5a - 1 = 24$
16. $10t - 5 = 95$
17. $5M + .4 = 2.9$
18. $11q = 27 + 2q$

46. Solution of Problems.—Most problems can be solved very simply by the literal equation. Moreover, the equation shows the connection between the various numbers in a problem, so that we learn the solution of this one problem and at the same time of all others like it. First, translate the English statement of the problem into the shorthand of the equation. Use an appropriate letter for the unknown number as i for **interest** or r for **radius**. Then solve the equation to find this unknown number and check the work by substituting the number found in the statement of the problem.

Twice the interest on a sum of money plus \$ 10 is \$ 38. What is the interest?

2 times interest	plus \$ 10	equals	\$ 38	(1)	
$2i$	$+$ \$ 10	$=$	\$ 38	(2)	
	$2i$	$=$	\$ 28	(3)	How?
	i	$=$	\$ 14	(4)	How?

Check: 2 times \$ 14 plus \$ 10 gives \$ 38. What is your conclusion?

EXERCISES

1. Seven times a number and 3 more equals 24. Find the number.

2. Eight times a number less 5 equals 27. Find the number.

3. The circumference of a circle is 44 ft. Find the radius.

4. The area of a triangle is 15 sq. ft. and the altitude is 6 ft. What is the length of the base?

5. The volume of a rectangular solid is 360 cu. ft. Find the altitude if the length is 15 ft. and the width 4 ft.

6. The area of a rectangle is 42 sq. in. and the width is 6 in. What is the length?

7. State in the shorthand of the equation the business expression, the amount, a, from money loaned equals the principal, p, plus the interest, i.

8. From the above statement find the interest if the principal is $ 478 and the amount is $ 532.

9. Find the principal if the interest is $ 79 and the amount is $ 664.

10. Express by the shorthand of the equation the business statement, the interest, i, equals the principal, p, times the rate, $\frac{r}{100}$, times the time, t, in years.

11. Use the equation from Ex. 10 and find what the interest will be on $ 200 at 6 % for 2 yr.

12. What principal gives $ 25 interest at 5 % in ½ yr.?

13. At what rate will $ 950 give $ 140 interest in 2 yr.?

14. In how long a time will $ 300 give $ 60 interest at 4%?

15. What will be the interest on $ 500 at 3.5 % in 4 yr.?

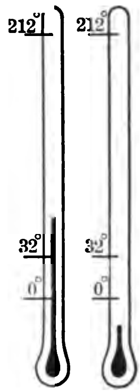
16. What principal gives $ 235.60 interest in 2 yr. at 6 %?

17. If Tom had $ 4 more than twice as much money as he has, he could buy a rifle costing $ 16. How much money has Tom?

18. Eight boys each put in the same amount of money to build a camp. The father of one of the boys gave them $ 25 toward it. They were then able to build a camp costing $ 145. How much did each boy put in?

19. Irene's weight increased by 18 lb. equals 120 lb. What is Irene's weight?

20. Find your weight and make similar problems for the class to solve.



47. Signed Numbers.—Sets of numbers frequently occur which are opposite in meaning. Such are north and south latitude; going east and going west; credit and debit; above and below zero on the thermometer; the pull of the two teams in a tug of war; above and below sea-level; money earned and money spent. The sign + is placed before one, called a **positive number,** and the sign − before the other, called a **negative number.** Thus, 45° north latitude is written + 45°, and 37° south latitude is written − 37°. Similarly, 25° above zero is written + 25°, and 13° below zero is written − 13°.

EXERCISES

Express the following as signed numbers:

1. 46° east longitude.
2. 52° west longitude.
3. 78° above zero.
4. A loss of $ 45.
5. A gain of $ 19.
6. $ 5 spent.
7. $ 17 earned.
8. $ 2.75 deposited in a bank.
9. $ 7 withdrawn from a bank.
10. 900 ft. above sea-level.
11. 200 ft. below sea-level.
12. 350 ft. down in a mine.
13. A balloon pulling up 9 lb.
14. A stone weighing 16 lb.

15. A boat going 12 mi. an hour down a stream; 6 mi. an hour up a stream.

16. Explain the temperatures in the table to the right.

17. Which is the highest? the lowest?

18. Omaha is how much below the highest? how much above the lowest?

U. S. Weather Observations
Temperature for
December 31, 1918

Chicago	34
Denver	− 2
Dodge City	0
Omaha	4
Phoenix	46
Winnipeg	− 20

48. Combining Signed Numbers.—Two signed numbers may be combined into one. Thus, if we call north positive and south negative, the distance one is from the starting-point due to two trips, one 7 miles north, then turning and going 10 miles south, may be expressed by

$$+ 7 \text{ mi.} + (- 10 \text{ mi.}).$$

Since the 10 miles south are more than the 7 miles north the sum is -3 miles, or 3 miles south of the starting-place.

If the first number in an expression is positive, the sign $+$ before it is omitted. Hence a number without a sign before it is always considered positive. The sign $+$ to denote addition is also omitted between signed numbers.

Thus, $+ \$12 + (- \$15) = - \$3$ is written

$$\$12 - \$15 = - \$3.$$

Similarly, $+ 12 + (+ 7) = + 19$ is written

$$12 + 7 = 19,$$

and $- 12 + (- 7) = - 19$ is written

$$- 12 - 7 = - 19.$$

EXERCISES

Express as signed numbers and find the sum:

1. \$ 35 deposited in the bank and \$ 15 withdrawn.

2. A thermometer reading 24 degrees above zero drops 30 degrees.

3. A boy walks 5 miles east, then walks 9 miles west.

4. An elevator is 16 stories above the main floor and goes down 10 stories.

5. What is the sign of the sum in adding two negative numbers?

6. What is the sign of the result in adding two positive numbers?

7. If you add a positive and a negative number, when is the sign of the sum positive? when is the sign negative?

Without pencil, find the sum of the following signed numbers:

8.	$-7-10$	**13.**	$-8+11$	**18.**	$-11+10$
9.	$-9+12$	**14.**	$11-8$	**19.**	$12-20$
10.	$-9+7$	**15.**	$15-4$	**20.**	$-17+17$
11.	$-15+4$	**16.**	$-4+18$	**21.**	$18-18$
12.	$-9-8$	**17.**	$-7-9$	**22.**	$9-19$

Write the sums of the following signed numbers:

23.	-49	**26.**	-56	**29.**	324	**32.**	-39.5
	42		-32		-422		48.2
24.	-65	**27.**	56	**30.**	672	**33.**	-67.3
	-36		-56		-242		49.5
25.	-88	**28.**	-426	**31.**	-345	**34.**	-34.2
	89		523		-492		34.2

35. The thermometer registers $+10°$. What will it register when the temperature rises $15°$? when it falls $18°$?

36. A boy is $ 5 in debt. Represent his money if he goes in debt $ 7 more; if he gets $ 15 and pays his debt.

37. In any business, money received less money paid out equals profits. State this by a formula. Can profits ever be a negative number? What are profits then called?

38. Mary raises chickens. One month she paid $ 17.50 for feed and improvements on her chicken-house, while she sold eggs for $ 10.40. What were her profits?

39. The next month her expenses were $ 7.85, while her sales were $13.90. What were her profits that month?

49. Sum of Several Signed Numbers.—In finding the sum of **several** signed numbers, find the sum of the positive numbers and then the sum of the negative numbers. Find the difference between these two sums and place before it the sign of the larger number. Thus,

$$45 - 36 - 12 - 8 = 45 - 56 = -\mathbf{11}.$$

EXERCISES

Express the following as signed numbers and find the sum:

1. Deposits at the bank of \$ 45, \$ 63, and \$ 82; withdrawals of \$ 38, \$ 42, \$ 73, and \$ 97.

2. A thermometer reads + 56°. There is then a drop of 28°, a rise of 19°, then a drop of 27°, and a rise of 37°. Find the reading of the thermometer.

3. Along the equator, in the picture, mark carefully the degrees of longitude as signed numbers. Along one of the meridians mark similarly the degrees of latitude.

4. A ship starts at north latitude 34° and sails south 65°, north 17°, and then south 27°. Express these latitudes as signed numbers and find the location of the ship as to latitude.

5. A ship starts at west longitude 84° and sails 47° east, next 17° east, then 7° west, and finally 18° east. Express these as signed numbers and find the longitude of the ship at end of the last sailing.

6. A man bought three houses for \$ 2560, \$ 3500, and \$ 1750. He sold them for \$ 2800, \$ 3000, and \$ 1900. Express these as signed numbers and show the total profit.

50. Terms.—Literal expressions containing neither the sign $-$ nor $+$, and the parts of an expression between the signs $-$ and $+$, are called **terms.** Such are $5m^2n^3$, also $2gh^3$ and $7rk^2$ in $2gh^3 + 7rk^2$. Expressions of only one term, as $3a^4b^2$, are called **monomials**; of two terms, as $5g^2h - 7mn^3$, **binomials**; of three terms, as $2a^2m - 3qr^3 - 7m$, **trinomials.** Expressions of several terms are called **polynomials.**

An expression, as $7ab^2 - 9a^2b + 5a^3b^2$, is read "seven a b square, minus 9 a square b, plus five a cube b square."

EXERCISES

1. Turn to the exercises on page 46 and pick out the terms in each expression. Name the coefficients and exponents. Read each expression.

Write the following in symbols:

2. Five times a raised to the fourth power, times b cube, plus 9 times a, times b square, minus c.

3. Three times n, minus x square, times y, minus z raised to the fourth power, plus x cube, plus 5 times z square.

4. t, plus u, minus 3 times r cube, plus 16.

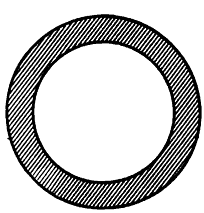
5. Two times m, plus 3 times n cube, minus 4 times m raised to the sixth power, plus 6, minus z square.

6. Five square meters, plus 17 square decimeters.

7. Eighty cubic meters, plus 450 cubic decimeters.

8. Twenty-five square Kilometers, minus 65 square Hektometers.

9. The area of a ring equals π times the square of the larger radius, minus π times the square of the smaller radius.



51. Combining Like Terms.—**Like terms** are terms which differ only in signs and coefficients, as $-3a^2b$ and $7a^2b$. Like terms are combined just as denominate numbers of the same denomination or other concrete numbers. Thus,

$$5mn^3 - 6mn^3 + 7mn^3 = 6mn^3, \text{ just as,}$$
$$5 \text{ bu.} - 6 \text{ bu.} + 7 \text{ bu.} = 6 \text{ bu.}$$

The only difference is that in combining like literal terms the final result may be negative. Thus,

$$8g^2h - 7g^2h - 4g^2h = -3g^2h.$$

EXERCISES

1. $-5ab + 3ab - 2ab + 12ab = ?$
2. $4x^3 - 9x^3 - 3x^3 + 2x^3 = ?$
3. $-8a^2bc + 9a^2bc - 4a^2bc + 7a^2bc = ?$
4. $-9kg^5 + 7kg^5 - 13kg^5 + 3kg^5 + 12kg^5 = ?$

Combine:

5.	6.	7.	8.
$-3t$	$-8t^2u$	$9abc^2$	$-3k^3z^2x$
$5t$	$-9t^2u$	$-5abc^2$	$8k^3z^2x$
$-9t$	$7t^2u$	$7abc^2$	$-5k^3z^2x$
$-7t$	$-5t^2u$	$-3abc^2$	$9k^3z^2x$
$6t$	$8t^2u$	$8abc^2$	$7k^3z^2x$

9.	10.	11.	12.
$5m^3n$	$6pq^2$	$-5y^2z$	$15qed^2$
$-4m^3n$	$5pq^2$	$9y^2z$	$17qed^2$
$7m^3n$	$-8pq^2$	$7y^2z$	$-9qed^2$
$-9m^3n$	$-3pq^2$	$-8y^2z$	$7qed^2$
$-2m^3n$	$2pq^2$	$-7y^2z$	$-6qed^2$
$3m^3n$	$-9pq^2$	$5y^2z$	$8qed^2$

13. In a number contest, correct results were marked $+10$. Incorrect results and exercises not attempted were marked -10. What was Katherine's total score if she had

the marks + 10, + 10, − 10, + 10, + 10, − 10, + 10, + 10, + 10?

Hold a number contest, using signed marks for the score.

52. Adding Polynomials.—In finding the sum of two or more denominate numbers, they are written so that those of the same denomination fall under each other in columns. This is shown below to the left, in which are added: 2 mi. 17 rd. 1 yd.; 19 mi. 28 rd.; 56 mi. 2 yd.

Literal numbers are added similarly by writing them so that like terms fall in columns, as shown to the right below, in finding the sum of: $2mn^2 + 17m^2 + m^3$; $19mn^2 + 28m^2$; $56mn^2 + 2m^3$.

2 mi.	17 rd.	1 yd.
19	28	
56		2
77 mi.	45 rd.	3 yd.

$$\begin{array}{llll} 2mn^2 & + 17m^2 & + m^3 \\ 19mn^2 & + 28m^2 & \\ 56mn^2 & & + 2m^3 \\ \hline 77mn^2 & + 45m^2 & + 3m^3 \end{array}$$

Note that the literal part of each term is repeated, but not the abbreviation of the denominate number. Negative terms may arise in literal sums which cannot happen in denominate numbers. Neither can a literal term be reduced to another literal number as in denominate numbers: 2 bu. 7 pk. make 3 bu. 3 pk.

53. Checking Literal Additions.—Both addition and subtraction may be checked by giving a numerical value to each letter. For convenience each letter can be given the value 1, then the above will become:

$$\begin{array}{lllcr} 2 & + 17 & + 1 & = & 20 \\ 19 & + 28 & & = & 47 \\ 56 & & + 2 & = & 58 \\ \hline 77 & + 45 & + 3 & = & 125 \end{array}$$

First add each row and place the sum to the right. Then add each column, which is made up of the coefficients of the terms. The sum of the column to the right of the equality signs will equal the sum of the numbers in the horizontal row below the line, if the work is correct. Where has this form of addition arisen before?

EXERCISES

Perform the following additions and check:

1. 3 yd. + 2 ft.
4 yd. + 1 ft.

2. $3y + 2f$
$4y + 1f$

3. 5 gal. + 3 qt.
3 gal. + 2 qt.

4. $5g - 3q$
$3g + 2q$

5. $6n - 4s$
$4n + 7s$

6. $8xv^2 + 4tr^3$
$-5xv^2 - 7tr^3$

7. $-2r^3 + 3r^2 - 7r;\ 9r^3 + 8r^2 - 2r.$

8. $-8m^2n - 7mn - 9m^2;\ -3m^2n + 2mn + 12m^2.$

9. $0a + 0b + 0c;\ -4a - 2b - 3c.$

10. $a + 2b + 3c;\ -4a + 2b - 9c + d.$

11. $3m^3;\ 2m^3 - 7m^2 + 8;\ 2m^3 - 4m^2.$

12. $-12f + 2i;\ 4f + 7i;\ 2f - 2i.$

13. $a + b;\ 2a + 3c;\ c + d;\ 3b - 4d.$

14. $a - s;\ -r - s;\ -r - s - t;\ 3r - 2s + 5t.$

15. $3a^4 + 2a^3 - 2a^2;\ 11a^2 - 9a;\ -7a^4 + 5a^2 - 6a;\ -a^3 + 5a.$

16. $-3.75a^3 - 0.25a^2 + 7;\ -2.25a^3 + 7.5a^2 - 0.3.$

17. $-a^4 + a;\ -a^4 + 3a^3;\ a^3 - 3a^4;\ a^2 + a - 9.$

18. $b^3 + b^2 + b + 1;\ b^3 + 3b^2 - 2b - 3;\ -4b^3 + 4b^2 - b - 9.$

19. $14tp^3k + 7b^2;\ 5tp^3k - 9b^2;\ -12tp^3k - 5b^2.$

20. $a + b;\ c + d;\ e + f;\ a + b + 3c + 4f.$

21. On Monday a grain dealer sold 845 bu. oats, 250 bu. corn, and 2540 bu. wheat; on Tuesday 326 bu. oats, 546 bu. corn, and 856 bu. wheat; on Wednesday 567 bu. oats, 758 bu. corn, and 968 bu. wheat. Express each day's sale as a literal expression and find their sum.

22. A stock dealer has on hand 26 cattle, 367 hogs, and 87 sheep. He buys 256 cattle, 178 hogs, and 129 sheep. He sells 19 cattle, 179 hogs, and 76 sheep. Express each statement in literal signed numbers and find their sum.

Hold a number contest on addition of signed numbers.

54. Subtraction of Signed Numbers.—Subtraction of signed numbers, just as the subtraction of other numbers, is the reverse of addition. Thus, to take $-2m$ from $5m$ means to find a number which added to $-2m$ gives $5m$. This is clearly $7m$. We may write this,

$$5m - (-2m) = 7m \qquad \text{or} \qquad \begin{array}{r} 5m \\ -2m \\ \hline 7m \end{array}.$$

EXERCISES

Solve the following without pencil:

1. What number added to $-6a$ gives $-8a$?
2. What number added to $-7b$ gives 0?
3. What number added to $-3mn$ gives $8mn$?
4. What number added to $4ac$ gives $-5ac$?
5. What number added to $5a^2c$ gives 0?
6. What number added to $5t$ gives $2t$?
7. From $-12a$ take $-9a$. Show that this is the same as, "Find the number which added to $-9a$ gives $-12a$." What is this number?

In this way give the other statements for each of the following exercises and solve:

8. From $8c$ take $-8c$.
9. From $-ab$ take $-4ab$.
10. From $4a$ take $9a$.
11. From $7c$ take $10c$.
12. From $-9ab^2$ take $-10ab^2$.
13. From $-3mn$ take $4mn$.
14. From $-8x^2y$ take $-2x^2y$.
15. From 0 take $5ac^3$.
16. With this idea of signed numbers, we can take a larger number from a smaller number or from zero. In which of the above exercises has this been done?

Perform the following subtractions:

17. $\begin{array}{r} 3abc \\ \underline{8abc} \end{array}$

18. $\begin{array}{r} 0ab \\ \underline{5ab} \end{array}$

19. $\begin{array}{r} 4xy \\ \underline{14xy} \end{array}$

20. $\begin{array}{r} -\ 5mn \\ \underline{-\ 9mn} \end{array}$

21. $\begin{array}{r} 0 \\ \underline{-\ 7a} \end{array}$

22. $\begin{array}{r} 12a^2b \\ \underline{a^2b} \end{array}$

23. $\begin{array}{r} -\ 8r^3s \\ \underline{-\ 2r^3s} \end{array}$

24. $\begin{array}{r} 2tu \\ \underline{9tu} \end{array}$

25. $\begin{array}{r} abc \\ \underline{-\ 9abc} \end{array}$

26. Write the above exercises, using parentheses; thus, $3abc - (+\ 8abc) = -\ 5abc$.

55. Subtraction of Polynomials.—In subtracting polynomials, first write the subtrahend under the minuend with like terms in the same column. Then subtract the like terms just as you subtracted monomials. Check as in addition except that the numbers in the columns are subtracted. The above would be as here shown. Explain.

$$\begin{array}{rrrr} 5pk^2 & +\ \ 8p^3k & & -\ \ 3k^3 \\ 2pk^2 & +\ 11p^3k & -\ 4k^2 & +\ \ 7k^3 \\ \hline 3pk^2 & -\ \ 3p^3k & +\ 4k^2 & -\ 10k^3 \end{array}$$

$$\begin{array}{rrrrcr} 5 & +\ \ 8 & & -\ \ 3 & = & 10 \\ 2 & +\ 11 & -\ 4 & +\ \ 7 & = & 16 \\ \hline 3 & -\ \ 3 & +\ 4 & -\ 10 & = & -6 \end{array}$$

EXERCISES

Carry out the following subtractions and check:

1. From $3a + 2b + 5c$ take $4a + b - 8c$.
2. From $-\ 6x^3 + 3x^2 - 2x + 7$ take $2x^3 + 8x^2 - x - 7$.
3. From $3a^2 + 5b^2$ take $2a^2 + b^2$.
4. From $6p^2 + 5p^3 + p$ take $-\ p^2 - 2p - 6p^3$.
5. From $0a + 0b$ take $3a + 2b$.
6. From $3t - 9$ take $2t + u - 7$.

7. From $7t^2 + 8tu + u^2$ take $9t^2 - 8tu + u^2$.

8. From $5.4m^3 - 7m^2 - 8$ take $8.7m^3 - 9.5m^2 - 7.2$.

9. From $a^3 - a^2 - a + 1$ take $2a^3 - 3a^2 + 2a - 6$.

10. From $61a^2 - 15b^2$ take $-9a^2 + 7ab - 3b^2$.

11. From $a + b$ take $c + d$.

12. From $-9a$ take $-7a + 2b - c$.

13. From $3pq$ take $3pq + 4p^2q^2 + q^3$.

14. From the sum of $2a + 3b - 4c$; $5a - 3c - 4b$; and $-2a - 5b + 4c$ take the sum of $-7a + 4b - 3c$; $-2a - 3b - 2c$; and $8a - 2b - 4c$.

15. To the sum of $5a^2b + 4ab^2$ and $-7a^2b - 2ab^2$ add $9a^2b + 7ab^2$ minus $3a^2b - 2ab^2$.

16. $3a - 2b + 4a - 5b + 3a - 7b - 4a = ?$

17. $2r - 3s + 4s - 9r - 5s - 9r = ?$

18. $14ab + 7ac - 4ab + 9ac - 10ac = ?$

19. $3x^2 - 4x^2 + 6x - 7x^2 - 8x + 9x - x^2 = ?$

20. $7x - 2y - 3x + 9y - 4x + 4 = ?$

21. $3a - 4a + 5a - 9a + 7a + 4a = ?$

22. At 10 o'clock the temperature less 5 degrees is 0. What is the temperature at 10 o'clock?

23. At 8 a.m. during a certain week a thermometer registers as follows: Mon. $-10°$, Tues. $12°$ higher than on Mon., Wed. $15°$ lower than on Tues., Thurs. $10°$ lower than on Wed., Fri. $8°$ higher than on Thurs. What did it register Fri. 8 a.m.?

24. Mary wishes to buy a book worth \$ 1.75 but lacks 35 cents. How much money has Mary?

Hold a number contest on addition and subtraction of polynomials.

56. Signs of Grouping.—All operations placed within **signs of grouping,** as parentheses ( ), bars | |, braces { }, brackets [ ], and under the vinculum $\overline{\qquad}$, are to be carried out first. For instance,

$$\begin{aligned} 37 - (18 \div 6 \times 5 + 7) &= 37 - (15 + 7) \\ &= 37 - 22 \\ &= 15. \end{aligned}$$

Similarly,

$$\begin{aligned} 23 + 3[42 \div 7 + 5] &= 23 + 3[6 + 5] \\ &= 23 + 3 \times 11 \\ &= 23 + 33 \\ &= 56. \end{aligned}$$

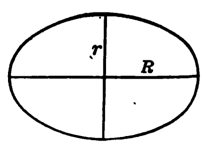
Signs of grouping occur frequently in formulas. Signs of grouping are also called signs of aggregation.

EXERCISES

Simplify each of the following:

1. $26 + (5^2 - 7 \times 3)$
2. $81 - [6^2 \div 18]$
3. $62 - 2[56 \div 7]$
4. $31 + \overline{42 \div 14 + 5}$
5. $[19 + 11] \div (21 \div 7 + 12)$
6. $(6 + 5 \times 2 - \sqrt{16}) - (5 \times 9 - 2 \times 4^2)$
7. $(13 - 3 \times 2 + 8 + 2) \times (5^2 - 2 \times 7)$
8. The circumference of an ellipse is given by the formula,

$$C = [r + R]\frac{22}{7}.$$



Find C when $r = 7$ cm. and $R = 9$ cm. $r = 9$ in. and $R = 1$ ft. 2 in.

9. In finding diameters of cog-wheels engineers use the formula,

$$D = (N + 2) \div P.$$

Find D when $N = 42$ and $P = 8$; $N = 46$ and $P = 7$.

10. Find the amount of $ 650 loaned at 6 % interest for 3 yr. In the formula t expresses the time in years and r the rate as a decimal. What are represented by A and p?

$$A = p[1 + rt]$$

11. Find the amount of $ 1225 for $3\frac{1}{2}$ yr. at 5 %.

12. Find the amount of $ 375 for 2 yr. 3 mo. at 5.5 %.

13. We shall soon find that the above formula can be changed to

$$A = p + prt.$$

To find how long it will require $ 740 to amount to $ 800 at 5 % substitute these known values in the last formula and solve for t.

14. How long will it take $ 1250 at 6 % to amount to $ 1500?

15. At what rate will $ 300 amount to $ 363 in 2 yr.?

16. At what rate will $ 550 amount to $ 650 in 3 yr.?

17. Fielding averages in baseball are computed by dividing the sum of assists and put-outs by the sum of assists, put-outs, and errors. Explain this statement by the formula,

$$F.Av. = [A + P] \div [A + P + E].$$

18. One summer Henry in playing baseball made 56 assists, 29 put-outs, and 7 errors. Find his average to three decimal places.

19. The same summer Kenneth made 38 put-outs, 8 assists, and 4 errors. Find his average.

Solve and check the following:

20. $7k + 3 = 2k + 18$

21. $11S - 7 = 5S + 17$

22. $9g - 5 = 3g + 7$

23. $8Q + 9 = 5Q + 18$

24. $17M - 11 = 5M + 49$

25. $6a - 5 = 4a + 7$

26. $9D + 2 = 3D + 5$

27. $14V - 4 = 8V + 5$

28. $12F + 7 = 9F + 11$

29. $35R + 19 = 15R + 63$

57. Multiplication of Signed Numbers.

$$3 \times 7k^2g = 21k^2g, \text{ just as,}$$
$$3 \times 7 \text{ bu.} = 21 \text{ bu.}$$

That is, $3 \times 7k^2g = 7k^2g + 7k^2g + 7k^2g = 21k^2g$. Just as, $3 \times 7k^2g$ means $7k^2g$ added 3 times, so, $3 \times (-7k^2g)$ means $-7k^2g$ added 3 times; that is, $-7k^2g - 7k^2g - 7k^2g = -21k^2g$. $-3 \times 7k^2g$ means to subtract $7k^2g$ three times; that is, $-(7k^2g) - (7k^2g) - (7k^2g)$, or $-7k^2g - 7k^2g - 7k^2g = -21k^2g$. $-3 \times (-7k^2g)$ means to subtract $-7k^2g$ three times, or $-(-7k^2g) - (-7k^2g) - (-7k^2g)$, and since $-(-7k^2g) = 7k^2g$, we have $7k^2g + 7k^2g + 7k^2g = 21k^2g$. That is:

$$3 \times 7k^2g = 21k^2g, \qquad -3 \times 7k^2g = -21k^2g,$$
$$3 \times (-7k^2g) = -21k^2g, \quad -3 \times (-7k^2g) = 21k^2g.$$

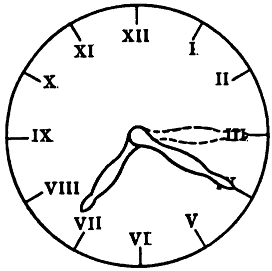
Hence, we have the following general rule: *The sign of the product of two positive or of two negative numbers is plus; the sign of the product of a positive and a negative number is minus.*

EXERCISES

Carry out the following multiplications:

1. $3 \times (-8a)$	**8.** $-4 \times (-10abc)$	**15.** $(-5)^2$
2. $-5 \times (-4b)$	**9.** $-7 \times 4ab^2$	**16.** $(-3)^2$
3. $-6 \times (-7a^2c)$	**10.** $3 \times (-9k)$	**17.** $(-25)^2$
4. $-2 \times (-7m^2n)$	**11.** $-5 \times (-4r)$	**18.** $(-9)^2$
5. $-9 \times 3r$	**12.** $7 \times (-9g^3)$	**19.** $(-21)^2$
6. $5 \times (-3y^2)$	**13.** $-8 \times 7r^2$	**20.** $(-6)^2$
7. $-4 \times (-7y^4)$	**14.** $-9 \times (-8y^3)$	**21.** $(-15)^2$

22. A miner goes down 100 ft. in a shaft, stops, goes up 60 ft. Express his position as a signed number. If he repeats this movement 4 times, express his position as a signed number.



23. A boy sold 3 rabbits which had been costing him 5 cents each per week. State his saving as a signed number.

24. State as a signed number the effect of turning the minute hand of a clock back 5 minutes three times.

25. A balloon takes in 8000 cu. ft. of gas. If 1000 cu. ft. have a lifting power of 65 lb., express its total lifting power as a signed number.

26. The balloonist puts in 7 sand-bags each weighing 25 lb. State this as a signed number and its effect upon the lifting power of the balloon.

27. If the balloonist throws out 2 bags, express the effect as a signed number.

28. The poll-tax in a county is \$ 2 for each man. State the effect upon the county treasury if 15 men move away.

29. If 12 men move into the county, state as a signed number the effect upon the county treasury. If 13 men move in and 25 men move away, state as a signed number the effect upon the county treasury.

In going north and south we call north + and south −.

30. Express as a signed number, a trip of 4 mi. north; also the distance gone in 5 such trips.

31. Express a trip of 7 mi. south as a signed number; also express the distance gone in 4 such trips.

32. Express the distance gone in 3 of these trips north and 5 trips south as a signed number. Show that it is $3 \times (+4) + 5 \times (-7) = ?$

58. Division of Signed Numbers.—Division, as heretofore shown, is the reverse of multiplication.

Since

$3 \times 7k^2g = 21k^2g$	3×7 bu. $= 21$ bu.
then, $21k^2g \div 7k^2g = 3$	21 bu. $\div 7$ bu. $= 3$
and $21k^2g \div 3 = 7k^2g.$	21 bu. $\div 3 = 7$ bu.

Also, since $3 \times (-7k^2g) = -21k^2g,$

then, $-21k^2g \div (-7k^2g) = 3.$ $\quad -21k^2g \div (-3) = ?$

Also, $-21k^2g \div 3 = -7k^2g.$ $\quad -21k^2g \div (7k^2g) = ?$

Similarly, since $-3 \times (-7k^2g) = 21k^2g,$

then, $21k^2g \div (-7k^2g) = -3$

and $21k^2g \div (-3) = -7k^2g.$

We have shown that,

$$21k^2g \div 7k^2g = 3,$$
$$21k^2g \div 3 = 7k^2g,$$
$$-21k^2g \div (-7k^2g) = 3,$$
$$-21k^2g \div 3 = -7k^2g,$$
$$21k^2g \div (-7k^2g) = -3,$$
$$21k^2g \div (-3) = -7k^2g.$$

If dividend and divisor have like signs, the quotient will be positive; if dividend and divisor have unlike signs, the quotient will be negative.

EXERCISES

1.	$36 \div -3 = ?$	7.	$-45r \div -5 = ?$
2.	$-32 \div -8 = ?$	8.	$-18t \div -18 = ?$
3.	$-8 \times 4 = ?$	9.	$42xy \div -7xy = ?$
4.	$-56a \div 7 = ?$	10.	$-18mn \div -3mn = ?$
5.	$49m \div 49 = ?$	11.	$-72a^2b^2 \div 8 = ?$
6.	$-5a \div -5 = ?$	12.	$-3a \div -3 = ?$

Find the value of the literal number in the following:

13.	$-3a = -18$	16.	$-12r = 36$
14.	$-4b = -12$	17.	$-a = 9$
15.	$5r = -10$	18.	$13t = -26$

19. George spent 18 cents. Express this as a signed number. If he spent this money in 6 days, how much was it per day?

20. Mary put 85 cents into the savings-bank. Express this as a signed number. If she earned the money in 5 days, how much did she average per day?

21. During a week corn decreased in price 24 cents. Express this as a signed number and find the average for each of the 6 days.

22. In a track-meet of 7 events the Reds made 41 points and the Blues 22 points. Express as a signed number how much the Reds were ahead and the average for each event.

23. Give the corresponding statements for the Blues.

24. Margaret spent 16 cents at the rate of 4 cents per day. Express each as a signed number. Find in how many days she spent the 16 cents.

25. By not turning out the lights in the basement at night Marvin loses 4 cents per night. The loss for a month was 28 cents. Express each as signed numbers. How many nights had he left the lights burning?

26. A thermometer changed from 5° to − 16°. Express the change as a signed number. If the change was − 3° per hour, in how many hours did the change take place?

27. A thermometer changed from 45° to 27° in 5 hr. Express the total change and the average per hour.

Hold a number contest, using multiplication and division of signed numbers.

59. Laws of Exponents in Multiplication and Division.—By definition we have learned that q^3 is $q \times q \times q$ and that q^2 is $q \times q$. Hence, $q^3 \times q^2 = (q \times q \times q) \times (q \times q) = q \times q \times q \times q \times q$. By the same definition this is q^5. Similarly, $a^2 \times a^6 = a^8$, and $m^2 \times m = m^3$. Explain each of the last two processes. Therefore, *the exponent of any letter in a product equals the sum of the exponents of that letter in the terms multiplied together.*

Since $a^2 \times a^4 = a^6$, then $a^6 \div a^2 = a^4$. That is, *the exponent of any letter in a quotient equals the exponent of that letter in the dividend less the exponent of that letter in the divisor.*

60. Laws of Multiplication and Division.—To multiply or to divide one term by another, three things must be found:

1. *Sign before the term.*
2. *Numerical coefficient.*
3. *Letters and their exponents.*

Thus, $-15a^4b^6 \div 3a^2b = -5a^2b^5$. How?
$-4m^3n^2 \times (-7mn^5) = 28m^4n^7$. How?

EXERCISES

Carry out the following operations:

1. $a^3 \times a^2$
2. $m \times m^3$
3. $ab \times ac$
4. $c^2d \times cd^2$
5. $-7f^3 \times 8f^2$
6. $-6R^2 \times 9R^3$
7. $(-5s^2)(3s)$
8. $(-7r^3)(-3r)$
9. $a^9 \div a^4$
10. $g^5h \div g^3h$
11. $8j^3k^5(-7jk^2)$
12. $h^6 \div h^3$
13. $(w^2v)(-8w^3v^5)$
14. $(-7x^2z)(-8x^3z^7)$
15. $-72s^5 \div 8s^3$
16. $-63t^5 \div (-9t^3)$
17. $-9x^3y^2z \div 3x^2y^2z$
18. $45c^4d^3e^2 \div (-9c^3d^2e)$

61. Short Literal Multiplication and Division.—Multiplication and division of polynomials by monomials are carried out by multiplying or dividing each term of the polynomial by the monomial. Thus,

$$\begin{array}{r} 2a^2 - 3ab + 5b^2 \\ -\ 3a \\ \hline -\ 6a^3 + 9a^2b - 15ab^2 \end{array} \qquad 3ab\overline{)\,-\ 27a^3b - 9a^2b^2 + 3ab} \atop \qquad -\ 9a^2 \quad -\ 3ab \quad +\ 1$$

just as, denominate numbers are multiplied and divided,

$$\begin{array}{r} \text{36 rd. 2 yd. 5 in.} \\ 2 \\ \hline \text{72 rd. 4 yd. 10 in.} \end{array} \qquad 3\overline{)\text{27 cwt. 81 lb. 15 oz.}} \atop \text{9 cwt. 27 lb. 5 oz.}$$

62. Checks.—Multiplication and division of literal numbers may be checked by substituting numerical values, other than 1, for each letter. If $a = 3$ and $b = 2$, the multiplicand will evaluate to 20 and the multiplier to -9. Their product is -180. Evaluate the literal product for these values and compare.

Using the same values, the dividend evaluates to -1764 and the divisor to 18. Their quotient is -98. Evaluate the literal quotient for these values and compare.

EXERCISES

First multiply, then divide the polynomial by the monomial. Check each result.

1.	$15m^3n - 6m^2n^2 + 12mn^3$	$+ 3mn$
2.	$25g^4 - 15g^7 + 35g^3$	$- 5g^3$
3.	$42w^3v^2 + 49wv^3 - 56w^4v^4$	$- 7wv^2$
4.	$9g^2h^2 - 18g^3h^3 + 45g^4h^4$	$- 9g^2h^2$
5.	$8x^3y^3 + 4x^4y^2 - 12x^5y^3$	$+ 4x^3y^2$
6.	$8a^4 - a^3 + 7a^2$	$- a^2$
7.	$9y^6 + 45y^4 - 6y^5$	$- 3y^3$
8.	$8r^4 - 12r^5 - 20r^7$	$- 4r^3$

63. Further Signs of Grouping.—To simplify expressions containing signs of grouping, first simplify the part within the signs of grouping if possible. Then remove signs of grouping by carrying out any indicated operation. Thus,

$$\begin{aligned} 5m^2n - 6mn^3 + 3[2m^2n + 7mn^3 - 8m^2n] &= 5m^2n - 6mn^3 + 3[7mn^3 - 6m^2n] \\ &= 5m^2n - 6mn^3 + 21mn^3 - 18m^2n \\ &= ? \end{aligned}$$

A sign of grouping preceded by a minus sign means that the quantity within is multiplied by minus one and added to what goes before; or merely subtracted from what goes before. Thus,

$$\begin{aligned} 5xy^3 + 11q^2r - 7t^3 - (3xy^3 - 5q^2r + 4t^3) &= 5xy^3 + 11q^2r - 7t^3 - 3xy^3 + 5q^2r - 4t^3 \\ &= ? \end{aligned}$$

EXERCISES

1. $5(3g^3h^2 + 7mn - 2g^3h^2 + 4mn) = ?$
2. $-3[7a^4b^2 + 3c^5d + a^4b^2 - 6c^5d] = ?$
3. $23w - (4w + 7w) = ?$
4. $7x - 9y - (5x - 11y) = ?$
5. $13q + 11r - 3(3q - 5r) = ?$
6. $8tr^5 - 7v^2w + 5\{tr^5 - 3v^2w\} = ?$
7. $8x^2 + 7x - 5 - [9x^2 - 4x + 17] = ?$
8. $19mn^3 + 7m^3n - 2(3mn^3 - 4m^3n - 2mn^3) = ?$
9. $3[2f^2g + 7fg^3] - 4(f^2g - 9fg^3) = ?$
10. $5\{3s^4e - 7az^3 + 2xv^2\} - 3(21s^7e^3 + 7s^3e^2 + 5az^3) = ?$
11. $3z^3 + 5x^2 - 2(z^3 - 3x^2) + 5(2z^3 - 4x^2) = ?$
12. $3q^2 - 7r^3 + (42q^5 - 35q^3r^3) \div 7q^3 - 11r^3 = ?$

64. Literal Long Multiplication.—The product of 327 × 24 is really the product of each number in 300 + 20 + 7 times each number in 20 + 4. Multiplying one polynomial by another is very similar as the product of each term in the multiplicand by each term in the multiplier is found first and then these partial products added.

$$\begin{array}{r} 300 + 20 + 7 \\ 20 + 4 \\ \hline 6000 + 400 + 140 \quad \\ 1200 + 80 + 28 \\ \hline 7848 \end{array}$$

$$\begin{array}{llll} 5x^2 + 2xy - 3y^2 & & & \\ 2x - 5y & & & \\ \hline 10x^3 + 4x^2y & - 6xy^2 & & \\ \quad\quad - 25x^2y & - 10xy^2 & + 15y^3 & \\ \hline 10x^3 - 21x^2y & - 16xy^2 & + 15y^3 & \end{array}$$

65. Checks.—Using the check of Art. 62, in which x is replaced by 2 and y replaced by 3, the multiplicand evaluates to 5 and the multiplier to -11. Their product is -55, as is also the evaluation of the product of the polynomials. What is the conclusion?

EXERCISES

Find the following products and check:

1. $(3a - 2)(2a - 5)$
2. $(2a - 3b)(3a - 4b)$
3. $(5r^2 - 3s)(2r - 3s)$
4. $(-7t^2 - 3u)(-2t - 3u^2)$
5. $(3x + 2y)(3x - 2y)$
6. $(-3a^3 - a^2)(-2a - 1)$
7. $(-7a^2 - 2)(3a^2 - a)$
8. $(-8a^3 - a^2)(-a + b)$
9. $(m + n)(a - b)$
10. $(2t + 4u)(3r - 2s)$
11. $(-5a^2 - 4b)(-5a^2+4b)$
12. $(7r^3s^3t^3 - 3)(2r^2s - 5)$
13. $(3r^3 - 2r^2 - 7r + 1)(2r^2 - 3r + 5)$
14. $(-7a^3 - 3a^2b + 4ab^2 - b^3)(-2a^2 - 3ab - b^2)$
15. $(4a^3b - 4a^2b^2 - 5ab^3)(4a^2b - 3ab^2 - b^2)$
16. $(a^4 - a^3b + a^2b^2 - ab^3 - b^4)(3a^2 - 2ab - b^2)$
17. $(-5g^3 - 4g^2 - 7g + 4)(-3g^3 - 2g^2 - 4g + 1)$
18. $(6h^2 - 7h + h^3 - 1)(h + 2h^2 + 2)$

Hold a number contest on multiplication of polynomials.

66. Long Division of Literal Numbers.—In dividing $7a^3 + 11 - 7a + 6a^3$ by $2a + 3$ the dividend and divisor must first be arranged in descending power of some letter, that is, they must be written so that the exponents of some letter are continually decreasing. Otherwise the work is very similar to long division with Arabic numerals. The form here shown is strongly recommended. Compare it with that used in dividing 6574 by 83.

$$
\begin{array}{r l}
 & 3a^2 - a - 2 \\
\cline{2-2}
2a + 3) & 6a^3 + 7a^2 - 7a + 11 \\
 & \underline{6a^3 + 9a^2} \\
 & -2a^2 - 7a \\
 & \underline{-2a^2 - 3a} \\
 & -4a + 11 \\
 & \underline{-4a - 6} \\
 & 17 \text{ rem.}
\end{array}
$$

$$
\begin{array}{r l}
 & a^2 + ab + b^2 \\
\cline{2-2}
a - b) & a^3 \qquad\qquad - b^3 \\
 & \underline{a^3 - a^2b} \\
 & a^2b \\
 & \underline{a^2b - ab^2} \\
 & ab^2 - b^3 \\
 & \underline{ab^2 - b^3}
\end{array}
$$

Places must be left for missing terms. Thus, $6a^3 - 8a + 3$ would be written $6a^3 + 0 - 8a + 3$. Division of $a^3 - b^3$ by $a - b$ explains how to proceed when some terms are missing.

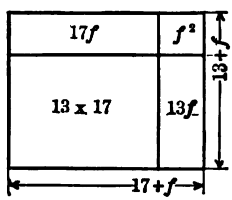
67. Checks.—Multiply the quotient by the divisor and add in the remainder to obtain the dividend just as in numerical checks. The work may be shortened by substituting numerals for the letters and carrying out the same checks. Let a in the first division be 2. The quotient is then 8. What is the divisor? their product? the sum of this product and the remainder? What is the value of the dividend if $a = 2$? What is the conclusion? Never use the numeral 1 or 0 in checking.

With two or more letters give each a different numerical value. Check the second division by letting $a = 3$ and $b = 2$.

EXERCISES

Perform the following divisions and check:

1. $(a^2 - 2ab + b^2) \div (a - b)$
2. $(-5m + m^2 + 6) \div (m - 2)$
3. $(6x^2 + x - 15) \div (2x - 3)$
4. $(3a^2 - 4ab + b^2) \div (3a - b)$
5. $(6m^2 - 17m + 12) \div (3m - 4)$
6. $(49 + 14x - 3x^2) \div (7 - x)$
7. $(28t^2 - 10tu - 2u^2) \div (-2t + u)$
8. $(-14r^2 + 25rs - 6s^2) \div (-7r + 2s)$
9. $(-4ab + 3a^2 + b^2) \div (3a - b)$
10. $(xy + 6x^2 - 15y^2) \div (3x + 5y)$
11. $(a^3 + b^3) \div (a + b)$
12. $(a^3 - b^3) \div (a - b)$
13. $(a^4 - b^4) \div (a^2 - b^2)$
14. $(a^5 - b^5) \div (a - b)$
15. $(a^5 + b^5) \div (a + b)$
16. $(27a^3 + 8) \div (3a + 2)$
17. $(27a^3 - 8) \div (3a - 2)$
18. $(8r^3 - 27) \div (2r - 3)$
19. $(8ab + b^2 + 7a^2) \div (a + b)$
20. $(3xy + 9x^2 + y^2) \div (3x + 2y)$
21. $(27a^3 - 5 + a^2 + 36a - 8) \div (3a - 2)$
22. $(ab - ad + bc - bd) \div (a - b)$
23. $(a^2 + 2ab + b^2 - c^2) \div (a + b - c)$
24. A rectangle 13 ft. by 17 ft. has each dimension increased f feet. Express its length and width and find its area after the increase.



25. Find the area of a circle whose radius is 4 ft. If the radius is increased to $(4 + f)$ feet, what is then the area? What is the increase in area?
26. The area of a rectangle is $a^2 + 11a + 28$ sq. in. Find the width if the length is $(a + 7)$ inches.

Hold a number contest on division of polynomials.

68. Equations Containing Signs of Grouping.—Equations containing signs of grouping often arise. In all such equations it is first necessary to remove the signs of grouping. Thus,

$$5m - 3(4 + m) = 14 \quad (1)$$
$$5m - 12 - 3m = 14 \quad (2) \quad \text{How?}$$
$$5m - 3m - 12 + 12 = 14 + 12 \quad (3) \quad \text{Why?}$$
$$2m = 26 \quad (4) \quad \text{How?}$$
$$m = 13. \quad (5) \quad \text{How?}$$

Check the root by substituting it in (1).

EXERCISES

Solve and check the following equations:

1. $6a + (5a - 9) = 13$
2. $3m + (m + 2) = 10$
3. $5b - (2b - 3) = 45$
4. $3a - (4 - 2a) = 11$
5. $7k - (2k + 3) = 37$
6. $8t - (3t + 4) = 61$
7. $15g - (8g + 5) = 30$
8. $(12r + 2) - 6 = 20$
9. $10A + 2(3A + 5) = 90$
10. $14B - 5(2B + 2) = 30$
11. $6(2a + 4) = 4a + 96$
12. $12y - (3y + 5) = 31$
13. $-(-5a + 2) - (3a - 7) = 13$
14. $18x - 6 = 4(3x + 9)$
15. $(2a - 5) + (3a + 2) = -3$
16. $2(-2k + 5) - 10 = 3k + 35$
17. $5z + (z - 10) = 2(z + 13)$
18. $9t + 3(2t + 7) = 186$
19. $8E - 3(2E + 5) = E + 1$
20. $8E - 3 - (2E + 5) = 76$

21. Find a number n such that if 13 less than the number be multiplied by 3 and 5 times the number be added to this, the result is 17. Set up the equation, solve, and verify.

22. Four times the sum of a number and 5 equals 32. Find the number.

23. If 7 be subtracted from a number and this remainder be multiplied by 3, the result will be 6. Find the number.

69. Equations.—Equations may contain signs of grouping, as $(m - 3)(m + 5)$. The signs of grouping are then removed by carrying out the indicated operation.

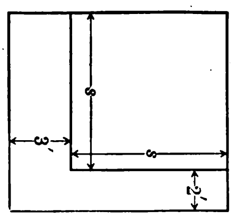
$$
\begin{array}{lll}
(m - 3)(m + 5) = (m + 1)(m - 7) & (1) & \\
m^2 + 2m - 15 = m^2 - 6m - 7 & (2) & \text{How?} \\
m^2 - m^2 + 2m + 6m - 15 + 15 & & \\
\quad = m^2 - m^2 - 6m + 6m - 7 + 15 & (3) & \text{Why?} \\
8m = 8 & (4) & \text{How?} \\
m = 1 & (5) & \text{How?}
\end{array}
$$

Check by substituting in (1).

EXERCISES

Solve the following equations and check:

1. $(h - 3)(h + 5) = (h + 1)(h - 7)$
2. $(a^2 - 9) - (a - 3)^2 = 6$
3. $(t^2 - 36) \div (t + 6) = 1$
4. $a^2 + 4(a + 4) = (a + 2)(a + 1)$
5. $7a - 2(3a + 4) = 16$
6. $-3t - 5(2t - 4) - 8t = -1$
7. $(m^2 - 6m + 9) \div (m - 3) = 4$
8. $(g + 5)(g - 5) + 2g = g^2 - 7$
9. $(2a + 3)(3a - 4) - (6a^2 - 2) = 0$



10. If one side of a square is increased 2 ft. and the other 3 ft., the area of the rectangle formed equals the area of the square and 36 sq. ft. more. Find the side of the square.

11. Express the length and width of a rectangle that is 8 ft. longer than it is wide. A new rectangle formed by taking 4 ft. from the length and adding 2 ft. to the width has an area equal to the original rectangle. Find the width.

III

FACTORS AND SPECIAL PRODUCTS

70. Factoring.—Finding those numbers which multiplied together produce a given number is called **factoring.** Thus, we factor 15 when we note that it is composed of 5×3. The 3 and 5 are the **factors** of 15.

Some literal numbers can be factored by finding a monomial factor of each term. Thus, $10a^2b^3 + 15a^3c$ has coefficients divisible by 5 and the literal part divisible by a^2 so that $10a^2b^3 + 15a^3c = 5a^2(2b^3 + 3ac)$. The factors of $10a^2b^3 + 15a^3c$ are then $5a^2$ and $2b^3 + 3ac$. Factoring is really the process of **undoing** multiplication.

71. Prime and Composite Numbers.—Numbers which have no factors are called **prime numbers.** Such are 11, 17, 31, $a + b$, $3c - 7m$, etc. Name some others.

Numbers which have factors are called **composite** numbers. Such are 12, 35, $5a^2c$, $3m^2 - 6mn$, etc. Name some others.

EXERCISES

Factor as many of the following as possible:

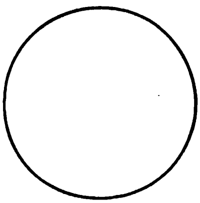
1.	21	**11.**	$12p^2 + 36$
2.	37	**12.**	$5a^2 - 35ab$
3.	135	**13.**	$8s^3z - 24sz$
4.	612	**14.**	$12m^4np - 36m^2$
5.	512	**15.**	$6a + 18b + 24c$
6.	225	**16.**	$2ab + 4ad - 8ac$
7.	343	**17.**	$2ab + 3cd + 4ef$
8.	$2a + 2b$	**18.**	$3x^4 - 24x^3 + 12x^2 - 21x$
9.	$3r + 24s$	**19.**	$a^2b^2c^2 + a^2b^2d^3 - a^2b^2e$
10.	$8r^2 + 4rq$	**20.**	$100k^3g^2 - 35k^4g^3 + 115k^7g^5$

72. Square Roots.—The **square root** of a number is **one** of its **two equal factors.** Since $2 \times 2 = 4$, the square root of 4 is 2. But since $(-2) \times (-2) = 4$, the square root of 4 is also -2. That is, $\sqrt{4} = 2$, or -2. Hence, a number has two square roots with opposite signs.

73. Quadratic Equations.—The area of a circle is found by replacing R with its numerical value in,

$$A = \frac{22}{7}R^2. \quad (1)$$

To find the radius of the circle whose area is $\frac{550}{7}$ it will be necessary to solve the equation,



$$\frac{22}{7}R^2 = \frac{550}{7} \quad (2)$$

$$R^2 = \frac{550}{7} \times \frac{7}{22} \quad (3) \quad \text{How?}$$

$$R^2 = 25 \quad (4) \quad \text{How?}$$

$$R = 5. \quad (5) \quad \text{How?}$$

Equations like (2), (3), (4) are called quadratic equations.

EXERCISES

1. Give the square roots of 81; 49; 1; 625; 144; 36.

2. $h = 5$, $h^2 = ?$ 3. $h = -5$, $h^2 = ?$ 4. $k = 7$, $k^2 = ?$ 5. $k = -7$, $k^2 = ?$

Find the two roots in each of the following:

6. $h^2 = 25$	9. $3k^2 = 48$	12. $5y^2 = 1.25$
7. $k^2 = 49$	10. $5s^2 = 20$	13. $100g^2 = 100$
8. $4m^2 = 36$	11. $13t^2 = 13$	14. $2b^2 = 2.88$

15. Make up three examples like the above for the other members of your class to solve.

Problems involving quadratic equations have two roots, but it often happens that one of the roots of the equation has no meaning or will not satisfy the conditions of the problem. For instance, in solving equation (4) under Art. 73 we could have $R = 5$ or -5, but a radius -5 has no meaning. In solving problems find all the roots, then reject those that do not have a reasonable meaning.

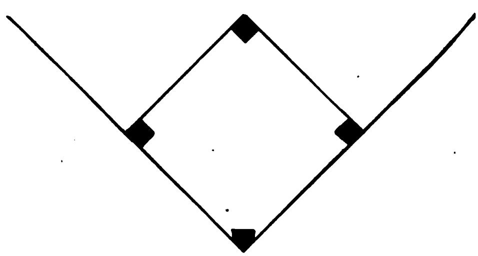
16. The area of a rectangle is 72 sq. in. The length is twice the width. Find its dimensions.

17. The area of a triangle is 32 sq. in. and the altitude is equal to the base. Find the base and the altitude.

18. Find the radius of the circle whose area is to be $\frac{352}{7}$ sq. ft.

19. A piece of land in the form of a square contains 40 A. Set up the equation, stating the length in rods squared equals the area in square rods. Then solve the equation to find the number of rods of fence necessary to enclose the land.

20. William stands on the right foul line 30 ft. back of first base. Show that the distance, D, he must throw to third base is given by the equation,



$$D^2 = 90^2 + 120^2.$$

Solve this equation for D.

21. John tied his cow to a stake by a rope so that the cow was able to graze over a circular plot having an area *of 1386 sq. ft.* What was the length of the rope?

74. Quadratic Equations.—

$$5L^2 - 3 = L^2 + 33 \qquad (1)$$
$$5L^2 - L^2 = 33 + 3 \qquad (2) \quad \text{How?}$$
$$4L^2 = 36 \qquad (3) \quad \text{How?}$$
$$L^2 = 9 \qquad (4) \quad \text{How?}$$
$$L = +3, -3; \text{ or } \pm 3. \qquad (5) \quad \text{How?}$$

Check by substituting both roots in (1). In solving quadratic equations always give the two roots.

EXERCISES

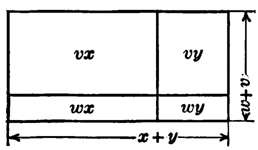
Find the two roots and check:

1. $6M^2 - 5 = 2M^2 + 11$
2. $5S^2 - 7 = 2S^2 + 20$
3. $2F^2 - 8 = 20 - 5F^2$
4. $1.23 + 2Q^2 = 3.15 - Q^2$
5. $2 + 7X^2 = 38 - 2X^2$
6. $5D^2 - .21 = .87 + 2D^2$

7. The area of a square increased by 9 sq. in. equals 90 sq. in. What is the dimension of the original square?

8. Three times the square of a certain number increased by 1,000 gives 1,075. Find the number.

9. Twice the square of a certain number decreased by 188 equals 100. Find the number.



10. Find the area of the rectangle that is $x + y$ long and $v + w$ wide.

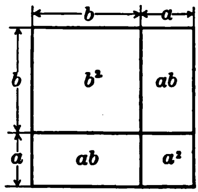
11. Perform the multiplications:

a. $(a + b)(a + b)$
b. $(c + d)(c + d)$
c. $(a + 3)(a + 3)$
d. $(a + 5)(a + 5)$
e. $(5 + a)(5 + a)$
f. $(r - s)(r - s)$
g. $(p - q)(p - q)$
h. $(a - 3)(a - 3)$
i. $(m - 5)(m - 5)$
j. $(5 - m)(5 - m)$

12. Make up at least six other such multiplications for the class to carry out.

75. Squares of Binomials.—By multiplication show that

1. $(a + b)^2 = a^2 + 2ab + b^2$
2. $(a - b)^2 = a^2 - 2ab + b^2$



Try writing the following without multiplying them and then multiply to see if your result is correct: $(r + s)^2$; $(m - n)^2$; $(t + 4)^2$; $(t - 2)^2$; $(a + 2b)^2$; $(2r + 3)^2$.

From the above examples we see that:

The square of the sum (or difference) of two numbers equals the square of the first term, plus (or minus) twice the product of the two terms, plus the square of the second term. Memorize examples 1 and 2 to use as formulas for squaring a binomial.

EXERCISES

Give the required results without the use of paper:

1. $(a + c)^2$
2. $(m + n)^2$
3. $(g - h)^2$
4. $(s + 2)^2$
5. $(b + 3)^2$
6. $(a + 4)^2$
7. $(r + 7)^2$
8. $(g - 5)^2$
9. $(3 - r)^2$
10. $(5 - b)^2$
11. $(2 + m)^2$
12. $(3r - 7)^2$
13. $(2a + 3b)^2$
14. $(7a - 2b)^2$
15. $(4t - 5u)^2$
16. $(3m + 8n)^2$
17. $(6m + 7n)^2$
18. $(3r^2 - 9)^2$
19. $(10y^2 - 4)^2$
20. $(3m^2 - 11)^2$
21. $(5a^2 - 3a)^2$
22. $(- 2t^2)^2$
23. $(- 3s)^2$
24. $(- 3kg)^2$

25. Show why the area of the above square whose sides are $a + b$ is $a^2 + 2ab + b^2$.

Solve the following equations and check:

26. $(3m + 2)^2 - 9m^2 = 16$
27. $3 - 16t^2 + (4t - 2)^2 = 39$
28. $(p + 5)^2 - 10p = 74$
29. $(3s - 2)^2 + 12s = 40$
30. $(2a + 5)^2 = 20a + 89$
31. $(4c - 3)^2 = 409-24c$

76. Formulas in Numerical Short Cuts.—Most numerical short cuts are based upon literal formulas. The formula of the last article can be used to make oral exercises out of seemingly long computations. Thus, $26 = 20 + 6$.

$$\begin{aligned} 26^2 &= (20 + 6)^2 \\ &= 20^2 + 2 \times 20 \times 6 + 6^2, \text{ or better} \\ &\qquad 20^2 + 6^2 + 2 \times 20 \times 6 \\ &= 400 + 240 + 36, \text{ or better } 400 + 36 + 240 \\ &= 676. \end{aligned}$$

Squares of numbers ending in 7, 8, or 9 are found in a manner similar to the following:

$$\begin{aligned} 49^2 &= (50 - 1)^2 \\ &= 2500 - 100 + 1, \text{ or better } 2500 + 1 - 100 \\ &= 2401. \end{aligned}$$

77. Squaring Numbers Ending in 5.—Any number ending in 5 may be represented by $10a + 5$. In 65, $a = 6$.

$$\begin{aligned} (10a + 5)^2 &= 100a^2 + 100a + 25 \\ &= 100a(a + 1) + 25. \end{aligned}$$

To square 65 we have $100 \times 6 \times 7 + 25 = 4200 + 25 = 4225$. That is, multiply the tens digit by one larger than itself and annex 25.

EXERCISES

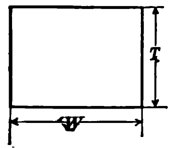
Find the required result as quickly as possible:

1.	32^2	8.	33^2	15.	38^2	22.	85^2
2.	19^2	9.	27^2	16.	83^2	23.	77^2
3.	16^2	10.	21^2	17.	37^2	24.	92^2
4.	17^2	11.	26^2	18.	66^2	25.	88^2
5.	18^2	12.	75^2	19.	48^2	26.	99^2
6.	25^2	13.	45^2	20.	53^2	27.	49^2
7.	65^2	14.	55^2	21.	57^2	28.	95^2

Hold a number contest on squaring numbers and solving equations.

29. State mentally the area of the following squares: 21 ft.; 39 m.; 49 in.; 51 cm.; 85 Km.; 71 yd.; 35 Dm.

30. In finding the strength of a beam builders use the square of its thickness. Find the squares of the following thicknesses: 25 in.; 48 cm.; 19 in.; 61 cm.



78. Factoring Trinomials.—In some trinomials the first and the third terms are perfect squares, while the second term is twice the product of the square roots of the first and the third terms. Such are $m^2 + 6mn + 9n^2$. Others may be written in this form. Thus, $h^4 + 25k^2 - 10h^2k$ can be written $h^4 - 10h^2k + 25k^2$. Each of these trinomials is derived from **squaring a binomial. Factoring** each of these **trinomials** is finding the binomial which squared gives the trinomial in question. The above are $(m + 3n)^2$ and $(h^2 - 5k)^2$. Check by squaring.

EXERCISES

Factor the following:

1. $m^2 - 6m + 9$
2. $h^2 - 10h + 25$
3. $s^2 - 6s + 9$
4. $t^2 + 14t + 49$
5. $16 + 40t + 25t^2$
6. $1 + 25m^2 - 10m$
7. $x^2 + 2xy + y^2$
8. $g^2 - 4gh + 4h^2$
9. $4a^2 + 12ab + 9b^2$
10. $49c^2 - 56cd + 16d^2$
11. $r^4 - 10r^2 + 25$
12. $9m^2 + 30mn + 25n^2$
13. $36t^2 + 25u^2 - 60tu$
14. $81r^4k^4 + 49 - 126r^2k^2$
15. $121k^2 + 25 - 110k$
16. $66ab + 121a^2 + 9b^2$
17. $1 - 2m^4 + m^8$
18. $25a^2b^2 - 40abc + 16c^2$
19. $144a^2 - 240ab^2 + 100b^4$
20. $9c^4 - 42c^2d^2 + 49d^4$

Whenever a monomial factor is found in each term, remove it first, then factor the other expression if possible. Thus,

$$a^3 + 6a^2 + 9a = a(a^2 + 6a + 9) = a(a + 3)^2.$$

21. $3a^2 + 6ab + 3b^2$
22. $4m - 8n + 12$
23. $4a^2 - 8ab + 4b^2$
24. $3t^3 - 6t^2 + 3t$
25. $-2k^2 - 4k - 2$
26. $12r^3 - 12r^2 - 12r$
27. $-a^3b + a^2b - 8ab^2$
28. $2h^2 + 20h + 50$
29. $r^3 + 10r^2 + 25r$
30. $3abcd - 24a^2b^2c^2d^2$
31. $2a + 4b + 8c - 16$
32. $3a^5 - 6a^4 - 18a^3 - 9a^2$
33. Show that the interest formula

$$a = p + prt$$

factors into the form

$$a = p(1 + rt).$$

79. Making Complete Squares.—On page 68 we learned how to square binomials. If we have two terms of the square of a binomial, we can find the third term. Thus, if we have $a^2 + 10ab$ and desire to find the number which added will make a complete square, we **divide** $10ab$ **by twice** the **square root** of the **first term**, $10ab \div 2a = 5b$, and square this resulting number. That gives $25b^2$ as the number to add. Then $a^2 + 10ab + 25b^2 = (a + 5b)^2$.

EXERCISES

1. $(a + ?)^2 = a^2 + 2ab + ?$
2. $(m + ?)^2 = m^2 + 2mn + ?$
3. $(r - ?)^2 = r^2 - 2rs + ?$
4. $(2a + ?)^2 = 4a^2 + 12ab + ?$
5. $(3r + ?)^2 = 9r^2 + 42r + ?$
6. $(2a + ?)^2 = 4a^2 + 20a + ?$
7. $(9r - ?)^2 = 81r^2 - 18r + ?$
8. $(2a - ?)^2 = 4a^2 - 12a + ?$
9. $(a - ?)^2 = a^2 - 8a + ?$
10. $(3a + ?)^2 = 9a^2 + 24a + ?$
11. $(5t - ?)^2 = 25t^2 - 30t + ?$
12. $(7a - ?)^2 = 49a^2 - 14a + ?$

In the following, supply the numbers that will make the trinomials perfect squares:

13. $a^2 + 2ab + ?$
14. $x^2 - 2xy + ?$
15. $r^2 + 6r + ?$
16. $4t^2 - 8t + ?$
17. $25k^2 - 50k + ?$
18. $16r^2 + 16r + ?$
19. $9s^2 - 42s + ?$
20. $36g^2 - 36g + ?$
21. $49h^2 - 42h + ?$
22. $12a + 4a^2 + ?$
23. $-18s + 9s^2 + ?$
24. $? - 24g + 36g^2$
25. $49m^2 - 28m + ?$
26. $100f^2 + 40f + ?$
27. $100f^2 + 60f + ?$
28. $100w^2 - 80w + ?$
29. $? - 36t + 81t^2$
30. $? - 70e + 25e^2$

Give the square root of each of the above trinomials after the square has been completed.

80. Quadratic Equations.—There are two kinds of quadratic equations. Those that contain only the second power of the unknown are called **incomplete quadratic equations.** This is the only kind we have studied so far. Those containing both the first and second power of the unknown are called **complete quadratic equations.** Thus,

$$a^2 - 6a = 40. \qquad (1)$$

n page 65 we solved incomplete quadratic equations and found two roots. Note that we found the square root of both members of the equation and still had a true equation. What other operations may we perform on both members of an equation and still have a true equation?

81. Solution of Complete Quadratic Equations.—To solve equations like $a^2 - 6a = 40$ we first find the number which added to $a^2 - 6a$ makes a perfect square as above. This is 9. We then add 9 to both members,

$$a^2 - 6a = 40 \qquad (1)$$

$$a^2 - 6a + 9 = 40 + 9 = 49 \qquad (2)$$

$$a - 3 = 7, \text{ or } -7 \qquad (3) \quad \text{How?}$$

$$a = 10, \text{ or } - \qquad (4) \quad \text{How?}$$

Check each result in (1).

EXERCISES

Find the roots for each of the following equations and check:

1. $a^2 - 6a = 16$
2. $t^2 + 8t = 9$
3. $b^2 - 12b = -32$
4. $m^2 - 2m = 8$
5. $g^2 + 12g = 64$
6. $e^2 - 18e = 0$
7. $k^2 - 6k = -5$
8. $4k^2 - 12k = 72$
9. $25m^2 - 30m = 40$
10. $L^2 - 20L = 21$
11. $4r^2 + 12r = 0$
12. $16q^2 + 32q = 9$

Some equations not in the above form can easily be made so by the use of one or more of the operations which give a true equation. Thus,

$$3a^2 + 2a - 1 = 0 \qquad (1)$$
$$3a^2 + 2a = 1 \qquad (2) \quad \text{How ?}$$
$$9a^2 + 6a = 3. \qquad (3)$$

Now complete and check as above.

EXERCISES

Solve and check the following:

1. $t^2 - 6t - 6 = 10$
2. $m^2 - 4m + 2 = 7$
3. $s^2 - 10s + 1 = 25$
4. $t^2 - 2t - 1 = 98$
5. $k^2 - 4k - 4 = -7$
6. $2k^2 - 8k - 4 = 6$
7. $3g^2 + 6g = 24$
8. $x^2 = 12x + 13$
9. $3t^2 - 2t + 5 = 6$
10. $6a = -a^2 + 16$
11. $6q^2 - 12q = 18$
12. $16a^2 - 24a = -8$
13. $2m^2 - 4m + 8 = 14$
14. $6p^2 - 4p + 10 = 12$
15. $r^2 - 12r + 2 = -30$
16. $7t^2 - 14t - 21 = 0$
17. $3a^2 - 2a = 1$
18. $5k^2 - 2k + 1 = 25$
19. $-2x^2 - 4x = -6$
20. $-z^2 - 6z = -16$

Hold a number contest on solving quadratic equations and finding binomial products.

If equations contain signs of grouping, as parentheses, these must first be removed by carrying out the required operation. This is shown in the following solution:

$$(x-4)^2+(x+2)^2 = 26 \qquad (1)$$
$$x^2-8x+16+x^2+4x+4 = 26 \qquad (2) \quad \text{How?}$$
$$2x^2-4x+20 = 26 \qquad (3) \quad \text{How?}$$
$$2x^2-4x = 6 \qquad (4) \quad \text{How?}$$

Complete the solution and check.

21. $(x-2)^2+(x-3)^2-2x = 3$

22. $(2a+1)^2-(a-1)^2 = 24$

23. $(3r+2)^2-(2r+3)^2 = 40$

24. $(m+2)^2+5m^2 = 4m+10$

25. $(m+4)^2+(m+3)^2 = m^2+1$

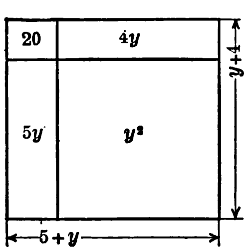
26. $3t^2+(2t+5)^2 = 6t^2+2t+65$

27. The square of a number added to three times the number equals 40. Find the number.

28. Twice the square of a number less 30 equals 4 times the number. Find the number.

82. Further Binomial Products.—Multiplication gives

$$(y+4)(y+5) = y^2+9y+20, \qquad (1)$$
$$(M+6)(M-4) = M^2+2M-24. \qquad (2)$$



Each product contains three terms: the first is the square of the literal term; the second is the literal term whose coefficient is the sum of the two numerical terms, as $4+5$ in (1) and $6-4$ in (2); the third term is the product of the numerical terms with the proper sign, as 4×5 in (1) and $6\times(-4)$ in (2).

EXERCISES

See how long it takes to give the following products:

1. $(r + 6)(r + 7)$
2. $(s - 7)(s + 2)$
3. $(r + 5)(r + 3)$
4. $(s + 2)(s + 9)$
5. $(g - 4)(g - 11)$
6. $(c - 5)(c + 2)$
7. $(t + 1)(t + 2)$
8. $(m - 4)(m + 4)$
9. $(m + 3)(m - 7)$
10. $(3a + 7)(3a - 7)$
11. $(ab - 2)(ab + 5)$
12. $(2k - 1)(2k + 3)$
13. $(xyz - 2)(xyz + 7)$
14. $(h^2 - 9)(h^2 - 7)$
15. $(mn - 7)(mn + 12)$
16. $(4a + 5)(4a + 5)$
17. $(r - 8)(r + 13)$
18. $(2v - 3)(2v + 1)$

Solve the following equations and check:

19. $(s - 5)(s + 2) = s^2 + 20$
20. $(r + 4)^2 - (r + 2)^2 = 24$
21. $(m - 9)(m - 5) + 4m = 36$
22. $(3 + r)(3 + r) + r^2 - 4r = 49$

23. If one side of a square is increased 3 ft. and the other 4 ft., give a mathematical statement for the area of the rectangle thus formed.

24. A picture 20 in. long and 14 in. wide has a frame whose width is expressed by W. Write a statement for the area of the frame.

25. A picture, including the frame, is 16 in. long and 9 in. wide. Write a mathematical statement for the area of the frame.

26. A circle is drawn with a radius r. Express the area of a circle whose radius is 2 in. greater.

83. Numerical Products.—Finding products of two numbers between 10 and 20 is a good application of the last multiplication formula. Such numbers are of the form $10 + a$ and $10 + b$. Explain.

$$(10 + a)(10 + b) = 100 + 10(a + b) + ab \qquad (1)$$

For 14×17,

$$(10 + 4)(10 + 7) = 100 + 10(4 + 7) + 4 \times 7 \qquad (2)$$
$$= 100 + 10(4 + 7 + 2) + 8 \qquad (3)$$
$$= 100 + 130 + 8 = 238. \qquad (4)$$

In practice find the product of the units, write the unit figure of this product, add any tens from this product to the sum of the unit's figures in the numbers, add 10 to this, and place the result before the number written.

Such products as 34×27 take the form

$$(30 + 4)(30 - 3) = 900 + 30(4 - 3) - 12 = ? \qquad (5)$$

EXERCISES

Carry out the following multiplications as far as possible without the use of pencil and paper:

1. 14×16	**8.** 32×29	**15.** 32×28	**22.** 15^2
2. 13×15	**9.** 23×17	**16.** 27×32	**23.** 25^2
3. 17×18	**10.** 24×21	**17.** 42×39	**24.** 75^2
4. 15×19	**11.** 24×26	**18.** 27×21	**25.** 41^2
5. 19×16	**12.** 56×54	**19.** 34×36	**26.** 32^2
6. 19×18	**13.** 38×42	**20.** 53×57	**27.** 45^2
7. 19×22	**14.** 23×17	**21.** 47×43	**28.** 52^2

29. Find the cost of 13 packages of dates at 17 ¢ each.

30. Find the area of the rectangle 12 ft. by 16 ft.

31. Find the area of the rectangle 68 ft. by 72 ft.

32. If one side of a square is increased 5 ft. and the other 3 ft., the area of the rectangle formed equals the area of the square plus 55 sq. ft. Find the sides of the square.

84. Factoring Trinomials of the Form $x^2 + px + q$.—In order to factor the trinomial $b^2 - 9b + 20$ we must find two factors whose product is 20 and whose sum is -9. These are -4 and -5. Hence,

$$b^2 - 9b + 20 = (b - 4)(b - 5).$$

In all factoring exercises some will be given which are reviews of cases previously studied.

EXERCISES

Factor the following:

1. $m^2 + m - 20$
2. $t^2 + 10t + 21$
3. $h^2 - 7h + 12$
4. $k^2 + 16k + 63$
5. $g^2 + 3g + 2$
6. $a^3 + 13a^2 + 22a$
7. $t^2 - 10t - 11$
8. $L^2 + 36 - 13L$
9. $y^2 - 5y - 36$
10. $2b^3 + 2b^2 - 40b$
11. $k^2 + 56 - 15k$
12. $k^2 - 56 - k$
13. $a^2 + 5ab + 6b^2$
14. $r^2 - 6s^2 - rs$
15. $y^2 - 12yk - 45k^2$
16. $4t^2 + 1 - 4t$
17. $tr^2 - 4tr + 4t$
18. $t^2 - t - 6$
19. $10ab + 2a^2 + 12b^2$
20. $3s^3 - 9s^2 - 30s$
21. $g^2h^2 - 36h^2$
22. $4 + x^2 - 4x$
23. $m^2n - 6mn + 9n$
24. $4t - 77 + t^2$
25. $a^2b - 20ab + 100b$
26. $g^4 - 7g^2 - 8$

Solve the following equations and check:

27. $2 - x(x + 3) + x^2 = 8$
28. $(a + 7)(a + 2) - 9a = 23$
29. $(m - 2)(m - 9) - (m - 3)^2 = -1$
30. $(s + 5)^2 + (s + 3)^2 - 20 = 0$
31. $(2r + 5)^2 = (r + 2)^2 + r^2 - 3$

EXERCISES

1. Butter is about 80 % fat. A farm hand needs 4.8 oz. of fat daily. If he gets $\frac{1}{4}$ of it in butter, how much butter does he need daily?

2. Eggs contain 10.5 % fat. How many ounces of egg will make 2.1 oz. of fat?

3. Milk from a certain cow tests 6.5 % cream. How much milk will it take to produce 2 qt. cream?

4. About 13 % of milk is water. How much water is there in a pound of milk? How much milk will it take to make 6 oz. of the other ingredients?

5. A commission man sold goods for Mr. Andrews, charging him 3 % of the selling price. If Mr. Andrews received $ 410.31, for what were the goods sold?

6. A number multiplied by itself and then increased by 8 equals 44. What is the number?

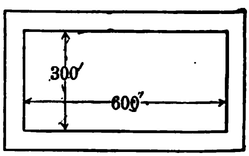
7. A number multiplied by itself and then increased by 7 equals 71. Find the number.

8. The square of a number added to 356 gives 500. What is the number?

9. A number squared, less twice the number, equals 99. Find the number.

10. The difference between two numbers is 12. Represent them. If their product is 108, what are they?

11. A city park is 300 ft. by 600 ft. Represent the area of a walk w ft. wide about the park.



12. Twice the area of a square decreased by 78 sq. ft. equals the area of a rectangle whose length is 3 ft. more and whose width is 5 ft. less than the sides of the square. Find the dimensions of the square and rectangle.

85. Product of Sum and Difference of Two Numbers.—By multiplication,

$$(a + b)(a - b) = a^2 - b^2.$$

From this,

$$\begin{aligned}(2cx^2 + 3bz^3)(2cx^2 - 3bz^3) &= (2cx^2)^2 - (3bz^3)^2 \\ &= 4c^2x^4 - 9b^2z^6.\end{aligned}$$

This formula can also be applied to finding products of numbers in which one is as much above a third number as the second is below the third number. Thus,

$$\begin{aligned}52 \times 48 &= (50 + 2)(50 - 2) \\ &= 2500 - 4 \\ &= 2496.\end{aligned}$$

EXERCISES

Use pencil and paper in the following only when it is absolutely necessary:

1. $(a + n)(a - n)$
2. $(r - s)(r + s)$
3. $(x + y)(x - y)$
4. $(2a - 3)(2a + 3)$
5. $(3m - 7)(3m + 7)$
6. $(4t + 2u)(4t - 2u)$
7. $(9c - 2d)(9c + 2d)$
8. $(2a^2 + 5b)(2a^2 - 5b)$
9. $(3r^2 - 7s)(3r^2 + 7s)$
10. $(5m^2 - 9m)(5m^2 + 9m)$
11. $(8x^2y^2 - 5)(8x^2y^2 + 5)$
12. $(6a + 11b)(6a - 11b)$
13. $(12tu - 10)(12tu + 10)$
14. $(7 - 11p^4)(7 + 11p^4)$
15. 51×49
16. 18×22
17. 57×43
18. 21×19
19. 17×23
20. 43×37
21. 23×17
22. 27×33
23. 52×48
24. 25×15
25. 34×26
26. 63×57

Solve the following equations and check:

27. $(3 + 2r)(3 - 2r) + 4r^2 + 2r = 19$
28. $25a^2 + (4 - 5a)(4 + 5a) = 4a$
29. $(t + 7)(t - 7) - 5t = t^2 + 1$
30. $(2a + 1)^2 + 5 = 14$
31. $(r - 5)(r + 5) - 2r = 23$
32. $(a + 3)(a + 2) + 3a = 15$

86. Factoring Differences of Two Squares.—The difference of two perfect squares can be factored into the product of the sum and difference of their square roots. Thus,

$$4m^2 - 49b^4 = (2m + 7b^2)(2m - 7b^2).$$

EXERCISES

Factor the following:

1. $m^2 - n^2$
2. $a^2 - b^2$
3. $x^2 - y^2$
4. $t^2 - 4$
5. $y^2 - 9$
6. $25 - a^2$
7. $16 - b^2$
8. $4m^2 - 49$
9. $36s^2 - y^2$
10. $100g^2 - 4a^2h^2$
11. $27a^2b^2 - 3$
12. $3t^2 - 3u^2$
13. $ar^2 - as^2$
14. $25mn^2 - m$
15. $3rg^2 - 27r$
16. $16a^2 - 8b^2$
17. $4k^4 - 4k^2$
18. $2r^3 - 16r^2 + 16r$
19. $a^2 - 2ab + b^2$
20. $x^2 - 6x + 9$
21. $5x^3 - 5x$
22. $h^2 - 9t^2$
23. $3a^2 - 12b^2c^2$
24. $2a^3 - 8ab^2$
25. $8a^3 - 32a$
26. $121a^3 - ab^2$
27. $144t^5 - t^3$
28. $15a^2b^2 - 60a^4b^4$
29. $a^3 + 2a^2b + ab^2$
30. $x^3 + x^2 + x$
31. $100a^2 - 25b^2$
32. $4a^2 + 12ab + 9b^2$
33. $2t^2u^2 - 288k^2$
34. $100a^2b^2 - 81c^2d^2$
35. $(A)^2 - (B)^2$
36. $(x - y)^2 - (6)^2$
37. $(x - y)^2 - (a + b)^2$
38. $(c - d)^2 - (r - s)^2$
39. $(a + b)^2 - 16$
40. $(m + n)^2 - 25$

Hold a number contest on squaring binomials and finding products of the sum and the difference of two numbers.

87. Product of Two Binomials in General.—By multiplication $(2a - 3b)(5a + 4b) = 10a^2 - 7ab - 12b^2$.

Here we notice:

1. The first term of the product is the product of the first term in each binomial.

2. The last term is the product of the last term in each binomial.

3. The middle term is the sum of the products of the first term in each by the second term in the other. That is,

$$2a \times 4b - 3b \times 5a = 8ab - 15ab = -7ab.$$

EXERCISES

Find the following products as far as possible without use of pencil and paper:

1. $(2a + 3)(a + 4)$
2. $(3r - 1)(5r + 7)$
3. $(t - 9)(5t - 3)$
4. $(2a - 7)(3a + 2)$
5. $(3m - 5)(5m - 2)$
6. $(3y + 1)(2y - 7)$
7. $(5t + 2)(4t + 3)$
8. $(4q - 2)(5q - 3)$
9. $(t + 7)(t - 7)$
10. $(t + 4)(t + 8)$
11. $(3r - 2)(2r + 3)$
12. $(5a - 4)(5a + 6)$
13. $(3a + 7)(3a - 4)$
14. $(5 + a)(2 + a)$
15. $(3a + 2b)(2a + 3b)$
16. $(4r - 7s)(2r - 6s)$
17. $(3mn - 4)(2mn - 3)$
18. $(10t - u)(2t + 5u)$
19. $(2ab - 3)(2ab + 4)$
20. $(2x + 3y)(3x + 4y)$
21. $(3m - n)(2m + 5n)$
22. $(ab - c)(ab - 3c)$
23. $(3r + 7s)(3r + 2s)$
24. $(3xy - 7m)(3xy + 7m)$
25. $(4r^2 - 9)(2r^2 - 3)$
26. $(3r^3 - 2r)(3r^3 + r)$
27. $(m - 7n^2)(m + 3n^2)$
28. $(g^2 + 5q)(2g^2 + 4q)$
29. $(8 - 3r^2)(2 - 7r^2)$
30. $(5rs - 9)(4rs + 15)$
31. $(a^2x - 4)(5a^2x - 9)$
32. $(3 - 2ab)(4 - 5ab)$
33. $(7 - 9a^2)(11 - 12a^2)$
34. $(4 - 3x^3y)(4 - 3x^3y)$

88. Factoring the General Trinomial.—In factoring $6a^2 - a - 15$, we first factor $6a^2$. This may be $3a \times 2a$ or $6a \times a$. That is, the factors of $6a^2 - a - 15$ may be

$$(3a + ?)(2a - ?) \text{ or } (3a - ?)(2a + ?),$$
$$(6a + ?)(a - ?) \text{ or } (6a - ?)(a + ?).$$

For the question marks must be placed either 3 and -5, -3 and 5, 15 and -1, or -15 and 1. When multiplied together these will give $6a^2$ for the first term and -15 for the third term but only one pair gives $-a$ for the middle term. That is,

$$(3a - 5)(2a + 3) \text{ for } 3 \times 3a + 2a \times (-5) = -a.$$

For $(6a - 5)(a + 3)$ the middle term is $13a$. Show this.

It is often necessary to try several times before the correct factors may be found. Always check the work by multiplication.

EXERCISES

Factor the following:

1. $3a^2 + 8a + 5$
2. $2a^2 - 5a + 3$
3. $4r^2 - 9r + 2$
4. $3a^2 + a - 2$
5. $2y^2 + 17y + 21$
6. $12m^2 + 9m - 3$
7. $10s^2 - s - 21$
8. $q^2 - 7q + 12$
9. $6g^2 - gh - 15h^2$
10. $ax^2 - ay^2$
11. $5ab + 3a^2 + 2b^2$
12. $4m^2 - 16m^2n^2$
13. $2a^2 + 6ab + 4b^2$
14. $x^3 + 4x + 4x^2$
15. $36x^2 - 49z^2$
16. $t^2 - 9t + 8$
17. $10v^2 - 6 + 28v$
18. $6z^2 + yz - 12y^2$
19. $20c^2 - 21d^2 - 13cd$
20. $7a^2 - 42an + 63n^2$

Make up exercises like the above for the other members of the class to factor. Do this by multiplying together two expressions like those on page 81.

Factor the following:

1. $a^2 + 13a + 36$	5. $a^2 - 9a - 36$
2. $a^2 - 5a - 36$	6. $a^2 + 20a + 36$
3. $a^2 + 5a - 36$	7. $a^2 + 16a - 36$
4. $a^2 + 15a + 36$	8. $a^2 - 16a - 36$

89. Numerical Applications.— Such multiplications as 6352×998, in which one factor is near 10, 100, 1000, etc., can be greatly simplified by writing the above $6352(1000 - 2)$.

$$\begin{array}{rl} 6352 & \\ 998 & \\ \hline 6352000 & = 6352 \times 1000 \\ 12704 & = 6352 \times 2 \\ \hline 6339296 & \text{subtracting.} \end{array}$$

EXERCISES

Use some short cut in each of the following:

1. 286×9	11. 57×63	21. 49^2
2. 1387×95	12. 54×99	22. 96^2
3. 275×98	13. 64×199	23. 84^2
4. 8544×997	14. 87×93	24. 95^2
5. 3765×98	15. 98×63	25. 55^2
6. 32×28	16. 69×98	26. 105^2
7. 43×37	17. 475×102	27. 53^2
8. 46×35	18. 674×97	28. 85^2
9. 4563×99	19. 745×98	29. 94^2
10. 857×98	20. 4506×998	30. 236^2

31. What is the cost of 1 doz. sweaters at \$ 5.98 each?

32. Find the cost of 1 doz. dresses at \$ 15.98 each.

33. Find the cost of 3 doz. ostrich plumes at \$ 8.98 each. What is the cost with a discount of 3 %?

34. Find the cost of 3 doz. pairs of shoes at \$ 4.98 a pair. What is the cost with a discount of 10 %?

35. Find the cost of 6 doz. overcoats at \$ 16.98 each. What is the cost with a discount of 8 %?

90. Factors by Grouping.—In factoring $b^3cA + m^2nA$, we notice that both terms contain A, so we may write $A(b^3c + m^2n)$ showing that A is multiplied by $b^3c + m^2n$.

Similarly $2b(r + s) + 5c(r + s)$ is factored into $(r + s)(2b + 5c)$. Here $(r + s)$ is common to the two expressions just as A was above. We may often group the terms of an expression so that each group will contain the same factor. Thus,

$$6am - 3an - 4bm + 2bn = 3a(2m - n) - 2b(2m - n)$$
$$= (2m - n)(3a - 2b).$$

Here we first took $3a$ out of the first two terms and $-2b$ out of the last two. Then $(2m - n)$ is common to the two expressions.

EXERCISES

Factor the following:

1. $ab + ac + rb + rc$
2. $am + an + bm + bn$
3. $2ay - 2az + 3by - 3bz$
4. $3r + 3s + mr + ms$
5. $5a^2 + 5b^2 + a^2y + b^2y$
6. $as - bs - ar + br$
7. $3x^3 + 3x^2 + 2x + 2$
8. $x^3 + x^2 + x + 1$
9. $3x^3 + 3x^2 + 3x + 3$
10. $x^3 - x^2 - x + 1$
11. $a^2 + ab + 2a + 2b$
12. $6pq + 9p + 4q + 6$
13. $4at^2 + 13at + 3a$
14. $9k^2 + 2 + 9k$
15. $4x^3 - 8x^2 - 4x + 8$
16. $a^3 - 5a^2 - 84a$
17. $r^2 - 24r + 95$
18. $12k^2 - 7k + 1$
19. $a^2 + 9b^2 - 6ab$
20. $a^2 + 2a + ab + 2b$
21. $36x^2y^2z^2 - 64k^2$
22. $t^2 + 18 - 11t$
23. $15g^2 + 6 + 19g$
24. $3r + 3s + 3a + 3b$
25. $2x^2y^2 - x^3y^3 + 2 - xy$
26. $x^2y^2z^2 - x^3y^3z^3 + 3 - 3xyz$
27. $3a^2b^2 + 6ab + 3abc + 6c$
28. $3ay + 4ax + 6by + 8bx$
29. $2t^2 - 18tu + 36u^2$
30. $6m^2n + 18mn + 8n^3$

Hold a number contest on factoring.

91. Another Method of Grouping.—Sometimes by grouping, expressions may be shown to be the difference of two squares. Thus,

$$\begin{aligned} a^2 + b^2 - 4 + 2ab &= a^2 + 2ab + b^2 - 4 \\ &= (a + b)^2 - (2)^2 \\ &= (a + b + 2)(a + b - 2). \quad \text{How?} \end{aligned}$$

Again, $t^2 + u^2 - a^2 - 2tu - 2ab - b^2$

$$\begin{aligned} &= t^2 - 2tu + u^2 - a^2 - 2ab - b^2 \\ &= (t^2 - 2tu + u^2) - (a^2 + 2ab + b^2) \quad \text{How?} \\ &= (t - u)^2 - (a + b)^2 \\ &= [(t - u) + (a + b)][(t - u) - (a + b)] \quad \text{How?} \\ &= (t - u + a + b)(t - u - a - b). \quad \text{How?} \end{aligned}$$

EXERCISES

Factor the following:

1. $x^2 - 2xy + y^2 - 9$
2. $r^2 - 2rs + s^2 - 16$
3. $t^2 - 4tu + 4u^2 - 9$
4. $4 - a^2 - 2ab - b^2$
5. $9 - r^2 - 4rs - 4s^2$
6. $a^2 + b^2 - x^2 - y^2 + 2ab - 2xy$
7. $m^2 - n^2 - x^2 + r^2 - 2mr + 2nx$
8. $9a^2 + 6ac + c^2 - 4b^2 - 4bd - d^2$
9. $4k^2 + g^2 - s^2 - 9r^2 + 4kg - 6rs$
10. $m^2 + 9n^2 - r^2 - 4s^2 - 6mn + 4rs$

Solve the following and check:

11. $5x^2 + 7 = 3x^2 + 25$
12. $(m - 7)^2 + (m - 7)(m + 7) + 15 = 51$
13. $(r + 3)(r - 3) + (r - 4)(r + 3) + r = 29$
14. $(2a + 3)(3a + 2) - (2a + 3)^2 + 2 = 0$
15. $(k + 4)^2 + (2k + 5)(k - 1) = (k + 4)(3k + 2)$

16. A picture is 3 in. longer than it is wide. Its area is 54 sq. in. Find its dimensions.

92. Factoring the Sum or Difference of Cubes.—Perform the indicated division:

1. $\dfrac{a^3 + b^3}{a + b}$	4. $\dfrac{8m^3 - n^3}{2m - n}$	7. $\dfrac{r^3 + 8s^3}{r + 2s}$
2. $\dfrac{a^3 - b^3}{a - b}$	5. $\dfrac{27s^3 - r^3}{3s - r}$	8. $\dfrac{125t^3 - 27k^3}{5t - 3k}$
3. $\dfrac{x^3 + y^3}{x + y}$	6. $\dfrac{27 + 8a^3}{3 + 2a}$	9. $\dfrac{8g^3 - 125m^6}{2g - 5m^2}$

A factor of a number is an exact divisor of the number. What then are the factors of the dividend in each of the above? Write the exercises with the dividend equalling the divisor times the quotient. Thus,

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2).$$

From these examples form a rule for factoring the sum of cubes; also the difference of cubes.

EXERCISES

Factor the following:

1. $r^3 + s^3$	8. $k^2 - 64$	15. $8a^3 + 27b^3$
2. $g^3 + h^3$	9. $k^3 + 64$	16. $27b^3 - a^3$
3. $r^2 - s^2$	10. $a^3b^3 + c^3$	17. $36x^2 - 49y^2$
4. $m^3 - 8$	11. $x^3y^3 - 27$	18. $3t^3 - 81$
5. $m^3 + 27$	12. $a^2b^2 - 16$	19. $5k^4 - 50k^2$
6. $a^3 - 8b^3$	13. $a^3b^3 - 125$	20. $64a^3 - 125b^3$
7. $k^3 - 27$	14. $a^3b^3 + 125$	21. $8m^6 - 27$

22. The difference of the squares of two consecutive numbers equals 15. What are the numbers?

23. If the cube of a number, increased by 27, be divided by the number plus 3, the result equals 3 times the number. Find the number.

93. Prime Factors.—An expression is not fully factored until it has been reduced to its **prime factors**; that is, to factors which cannot be factored further. Thus, $a^4 - b^4$ was not in its prime factors when factored into

$$(a^2 + b^2)(a^2 - b^2), \text{ because } a^2 - b^2 = (a + b)(a - b).$$

Hence, $a^4 - b^4 = (a^2 + b^2)(a + b)(a - b)$.

Again,
$$a^6 - b^6 = (a^3 + b^3)(a^3 - b^3)$$
$$= (a + b)(a^2 - ab + b^2)(a - b)(a^2 + ab + b^2).$$

Factors are needed a great deal in the solution of equations and in fractions which will be studied later. The prime factors are always desired. Hence, in factoring an expression, be constantly on the lookout to see that no factor can again be factored. Always check the work. How?

Make a list of all of the forms of factoring studied. Use it in the exercises below.

EXERCISES

Find the prime factors of the following and check:

1. 234
2. 90
3. 1260
4. 480
5. 4200
6. $h^2 - 25$
7. $7g^2 - 28$
8. $125 + 8f^3$
9. $500 - 4r^3$
10. $a^3b - 25ab^3$
11. $5a^4 - 36ab^3$
12. $7d^2 - 175t^4$
13. $4a^2 - c^2$
14. $6m^3 - 6n^3$
15. $7q^4 - 7r^4$
16. $w^6 - w^4v^2$
17. $w^6 - 64$
18. $r^3 - 125q^6$
19. $2u^3 - 16$
20. $125 - 5y^4$
21. $216 - 6q^4$
22. $w^2 + v^2 - 2vw$
23. $(u + i)^2 - 4u^2i^2$
24. $[s^2 + r^2]^2 - 4r^2s^2$
25. $a^2 - 4ab^2 + 4b^4$
26. $x^2 - 3x - 28$
27. $g^4 + 10g^2 + 16$
28. $k^4 - 2k^2 - 63$
29. $z^2 + 16z + 63$
30. $2t^2 - 5t - 12$
31. $v^4 - 2v^2 - 99$
32. $10c^2 - 11c + 3$
33. $m^4 + 13m^2 + 36$
34. $s^2 + q^2 - 25 - 2qs$
35. $10g^2 - 6k^2 - 11gk$
36. $z^2 + 24 - 10z$

Hold a number contest on factoring.

94. Equations Solved by Factors.—

$2 \times 0 = ?$ $3 \times 0 = ?$ $5 \times 0 = ?$ $n \times 0 = ?$ $5n \times 0 = ?$ Any number $\times\ 0 = ?$

If the product of two numbers is 0, what must one of them be?

$5a = 0$	$6t = 0$	$7m = 0$	$8r = 0$
$a = ?$	$t = ?$	$m = ?$	$r = ?$

We may solve an equation by factors. Thus,

$$5a - 10 = 0 \quad (1)$$

$$5(a-2) = 0 \quad (2) \qquad \text{Factoring (1).}$$

$$a - 2 = 0 \quad (3)$$

$$a = 2. \quad (4)$$

If the product of two numbers is zero, one of them must equal zero, but 5 cannot equal zero.

Again, if $a^2 - 9 = 0$

then $(a+3)(a-3) = 0$

$a - 3 = 0$ or $a + 3 = 0$

$a = ?$ or $a = ?$

Here we have two roots. Check them. What kind of equations have two roots?

Hence, to solve an equation by factoring:

1. Make one member equal zero. How?
2. Factor and set each literal factor equal to zero.
3. Solve the equations thus formed.

EXERCISES

Solve the following and check:

1. $7(r + 2) = 0$
2. $a(a - 2) = 0$
3. $m(m + 7) = 0$
4. $(m - 2)(m + 2) = 0$
5. $(m - 2)(m + 3) = 0$
6. $m^2 - 7m + 12 = 0$
7. $x^2 - 6x + 8 = 0$
8. $r^2 - 5r + 6 = 0$
9. $m^2 - 4m + 4 = 0$
10. $h^2 + 3h - 10 = 0$

Solve the following by factoring if possible:

11. $x^2 + x = 0$
12. $x^2 + 3x = 18$
13. $y^2 - 4y = 12$
14. $-6k = -k^2 + 27$
15. $y^2 + 5y - 50 = 0$
16. $t^2 - 2 = 7t + 76$
17. $k^2 + 88 = 26k + 19$
18. $y^2 + 4y - 7 = 70$
19. $s^2 + 50 = 14s + 5$
20. $9t - 6 = -42t^2$
21. $u^2 - 5 - 7u = 55$
22. $R^2 - 4R = 0$
23. $t^2 - 12t = 7t - 78$
24. $x^2 + 5x = x + 45$
25. $(r + 2)^2 - (r + 3)^2 = 17$
26. $(2k + 4)^2 - (2k + 1)^2 = 95$
27. $(2m + 3)^2 - (m + 4)^2 = -8$
28. $2z^2 - 5z + 3 = 0$
29. $5c^2 + 6c = 8$
30. $q^2 - 7q - 78 = 0$

Solve the following by completing the square:

31. $a^2 - 4a = 12$
32. $3t^2 + 2t = 33$
33. $4n^2 = 8n + 140$
34. $2k^2 - 4k + 5 = 35$
35. $3b + b^2 = b + 8$
36. $10j = -j^2 + 75$
37. $14R = 32 - R^2$
38. $8m^2 = 64 + 16m$
39. $-x^2 - 6x = -16$
40. $-z^2 + 4z = -96$

41. The sum of two numbers is 7 and the sum of their squares is 29. What are the numbers?

42. One of two numbers is 3 times the other. If their product is 75, what are the numbers?

43. The sum of the squares of two consecutive even numbers is 100. What are the two numbers?

44. The altitude of a rectangle is 20 % less than the base. If the area is 64.8 sq. ft., what are the dimensions of the rectangle?

45. Two men start from the same place, one going north and the other south. The first travels twice as fast as the second. Express the distance they will be apart in 1 hr. If they are 360 mi. apart in 5 hr., how many miles does each man travel per hour?

95. Several Signs of Grouping.—Several signs of grouping may occur in one expression, some of which are within the others. To simplify such expressions by removing the signs of grouping, begin by removing the innermost one first. This is shown in the following:

$$5a + 2[3a - b - 2(a - 3b) + (2a - 4\{a - 5b\})]$$
$$= 5a + 2[3a - b - 2(a - 3b) + (2a - 4a + 20b)]$$
$$= 5a + 2[3a - b - 2a + 6b + 2a - 4a + 20b]$$
$$= 5a + 2[- a + 25b]$$
$$= 5a - 2a + 50b = ?$$

EXERCISES

1. Explain the above processes very carefully.

Remove the signs of grouping and simplify:

2. $7m + 2n + [3m - (2n - 4m)]$
3. $3b - c - \{2c - [3b - 4c]\}$
4. $2w - (3w + [2w - 3v]) + 5v$
5. $4x - 7y - [2x - 3(y - 5x)]$
6. $- 3[2c - d] - (- 4c - 2[5c - 6d] + 7c)$
7. $2(3q - 2r + 4[r - 3q] - 5q + 3r)$
8. $3f - 2(g - 3[2f - 3g] + 5f - 2g)$

Solve and check the following equations:

9. $2[3m - 5] = 32$
10. $3(5h + 4) = - 28$
11. $- [3f - 7] = 16$
12. $- 2(5 - T) = 3T$
13. $- (4y - 9) = 2y + 6$
14. $3[k - 2(k + 5)] = - 2(11)$
15. $- (2e - 2[3 + 2e]) = - 4$
16. $[m - 4][m + 7] = m^2 + 3$
17. $[k - 5][k + 9] = k^2 - 27$
18. $3(g + 5)(2g - 7) = 6g^2 - 15$
19. $2(m - 8)(m + 7) = m^2 - 14m - 4$
20. $[z - 3][2z - 1] = z^2 + z + 12$
21. $3[2k - 1][k + 5] = 2k^2 + 7k + 9$

IV

COMMON FRACTIONS

96. Numerical Fractions.—What are the **numerator**, **denominator**, and **terms** of $\frac{2}{3}$? of $\frac{5}{12}$? of $\frac{4}{5}$? How many inches in $\frac{1}{3}$ of a foot? in $\frac{2}{3}$ of a foot? in $\frac{1}{9}$ of a yard? in $\frac{2}{9}$ of a yard? in $\frac{5}{9}$ of a yard? Show from this that the denominator tells the kind of a fraction; **names the fraction.** Also show that the numerator tells how many there are of the parts named by the denominator; it **numbers the fraction.**

Three inches is what part of a foot? of a yard? Six inches is what part of a foot? of a yard? A nickel is what part of a quarter? of a half dollar? of a dollar?

97. Equivalent Fractions.—Which is more, $\frac{2}{3}$ of a foot or $\frac{4}{6}$ of a foot? Show why $\frac{2}{3} = \frac{4}{6}$. Fractions as $\frac{2}{3}$ and $\frac{4}{6}$ are called **equivalent fractions.** Name two fractions that are equivalent to each of the following: $\frac{2}{3}$, $\frac{4}{8}$, $\frac{3}{15}$, $\frac{6}{12}$, $\frac{8}{12}$, $\frac{9}{36}$, $\frac{4}{7}$, $\frac{3}{18}$. Note that multiplying or dividing both terms of a fraction by the same whole number gives an equivalent fraction.

How many $\frac{1}{8}$ in $\frac{1}{4}$? in $\frac{1}{2}$? in 1? How many $\frac{1}{6}$ in $\frac{1}{2}$? in $\frac{1}{3}$? in 1? How many $\frac{1}{10}$ in $\frac{3}{5}$? in $\frac{1}{5}$? in $\frac{1}{2}$?

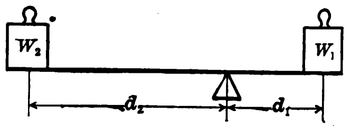
98. Fractions a Form of Division.—A fraction may be thought of as the numerator divided by the denominator. Thus, $12 \div 3 = 4$, while the fraction $\frac{12}{3}$ also equals 4 by dividing both terms by 3.

Write as fractions: $15 \div 5$; $36 \div 9$; $36 \div 7$; $42 \div 21$; $19 \div 5$.

99. Literal Fractions.—Literal fractions are of frequent occurrence, especially in formulas. They are always to be considered as indicated divisions. Thus,

$$\frac{15a^3b^2}{3a^2b} = 5ab; \quad \frac{17m^4n^5}{5c^2d} = 17m^4n^5 \div 5c^2d.$$

In the common teeter board, if the weights and lengths are as shown in the picture,



then,

$$W_2d_2 = W_1d_1. \quad (1)$$

From which,

$$W_2 = \frac{W_1d_1}{d_2}. \quad (2)$$

Evaluate the above if $W_1 = 60$, $d_1 = 8$, and $d_2 = 6$.

EXERCISES

1. State as a formula that the width of a rectangle equals its area divided by its length.

2. How can the depth of a rectangular solid be found from its volume, width, and length? State as a formula.

3. The depth in feet to which a rectangular coal bin must be filled with soft coal equals the number of tons multiplied by 35, and this divided by the length times the width each in feet. State as a formula.

4. To what depth must a bin that is 8 ft. long and 5 ft. wide be filled with soft coal to hold 4 T.? to hold 7 T.?

5. State as a formula the number of tons of soft coal held by a rectangular bin of any length, width, and depth.

6. Express as a formula that the price of one article, p, equals the price of a number of articles, P, divided by the number of articles, n.

7. Express the number of articles bought in terms of the total price and the price of one article.

8. Paper-hangers' estimate of amount of paper required for sides of a room: Find the perimeter of the room, divide by 1.5, subtract twice the number of doors and windows (openings), divide by 5 to get the number of double rolls. Show that this is expressed by the formula,

$$R = \left\{ \frac{2[l + w]}{1.5} - 2 \times O \right\} \div 5.$$

The number of single rolls is expressed by

$$r = 4 \left\{ \frac{[l + w]}{1.5} - O \right\} \div 5. \text{ How?}$$

9. Turn to page 192 and find the number of rolls of paper required for each of any two of the rooms.

10. Measure your schoolroom and find how many rolls of paper are necessary to paper above the boards. These paper strips will be about the length of those in the average room and so the above formula can be used.

11. Find the cost to paper your schoolroom at the price of paper and paper-hanging in your city.

12. The average of any team playing games equals the number of games won divided by the sum of the number won and the number lost. Express as a formula.

13. Find the average of the football team which has won 6 and lost 4 games; won 5 and lost 7 games.

14. The diameter of a tree or other round object is found by dividing the distance around by π. Express this as a formula. Use this formula to find the diameter of some large tree.

100. Signs of Fractions.—Both $-a \div b$ and $a \div (-b)$ will give the same negative quotient. Why? Hence,

$$\frac{-a}{b} = -\frac{a}{b} = \frac{a}{-b}; \qquad \frac{-12}{3} = -4 = \frac{12}{(-3)}.$$

101. Fractions to Lowest Terms.—Fractions whose numerator and denominator have no common factor are said to be in their **lowest terms.** All final fractions should be reduced to their lowest terms. Thus, $\frac{15}{24} = \frac{5}{8}$. How? Similarly $\frac{7a^3b^2}{5ab^3} = \frac{7a^2}{5b}$. By what were both terms divided?

By factoring and then dividing both terms,

$$\frac{a^2 - 4b^2}{a^2 - 5ab + 6b^2} = \frac{(a+2b)\cancel{(a-2b)}}{\cancel{(a-2b)}(a-3b)} = \frac{a+2b}{a-3b}.$$

EXERCISES

Show that the following equalities are true:

1. $\frac{-6}{2} = -3 = \frac{6}{-2}$
2. $\frac{-3}{9} = \frac{-1}{3} = \frac{3}{-9}$
3. $\frac{-32}{96} = \frac{-1}{3} = \frac{3}{-9}$
4. $\frac{-45ab}{90ab} = \frac{45ab}{-90ab}$
5. $\frac{-3xy}{a+b} = \frac{3xy}{-a-b}$
6. $\frac{3}{4} = \frac{27}{36} = \frac{27a}{36a}$
7. $\frac{3}{a-b} = \frac{3a+3b}{a^2-b^2}$
8. $\frac{m}{m-n} = \frac{m^2+mn}{m^2-n^2}$
9. $\frac{-38}{-19} = 2 = \frac{38}{19}$
10. $\frac{75}{-25} = -3 = \frac{-75}{25}$
11. $\frac{35}{-45} = -\frac{7}{9} = \frac{7}{-9}$
12. $\frac{-3a-3b}{-3} = a+b$
13. $\frac{-3a+6b}{-3} = a-2b$
14. $\frac{1}{x+2} = \frac{(3x+9)}{3(x+2)(x+3)}$
15. $\frac{a}{m-n} = \frac{5am+5an}{5(m-n)(m+n)}$
16. $\frac{3}{2(a+5)} = \frac{3a-9}{2(a+5)(a-3)}$

Reduce the following fractions to their lowest terms:

17.	$\frac{36}{48}$	**23.**	$\frac{7x^2}{14x^3}$	**29.**	$\frac{4a^2b - 4ab^2}{8a^2}$
18.	$\frac{124}{496}$	**24.**	$\frac{13m^2n}{39mn^2}$	**30.**	$\frac{a+b}{a^2-b^2}$
19.	$\frac{393}{684}$	**25.**	$\frac{45a^2br}{54a^3b^2}$	**31.**	$\frac{c^2-d^2}{c^2+2cd+d^2}$
20.	$\frac{105}{245}$	**26.**	$\frac{12a^2x^2y^2}{18axy}$	**32.**	$\frac{x^2-16}{x^2+x-20}$
21.	$\frac{3a}{6}$	**27.**	$\frac{42mnr}{35m^2r^2}$	**33.**	$\frac{x^2+5x+6}{x^2+7x+12}$
22.	$\frac{5mm}{6mn}$	**28.**	$\frac{3a+3b}{6}$	**34.**	$\frac{m^2+2mn+n^2}{m^2-n^2}$

35. $\frac{g^2-h^2}{g^3-h^3}$ **36.** $\frac{r^2+4r+4}{3r^2+11r+10}$

37. If L, W, and H are given in feet, then,

$$C = \frac{LWH}{4 \times 4 \times 8}$$

states the number of cords of wood in a rectangular pile. Find the number of cords in a pile 6′ by 10′ by 16′.

38. Machinists use the following formula:

$$E = \frac{4\pi d^2rV}{2\pi d^2l} + \frac{4\pi dlrV}{4rd^2l}.$$

Simplify each of the fractions.

39. The formula for the volume of a cone may be:

$$V = \frac{\pi R^2H}{3} \quad (1)$$

or

$$V = .2618D^2H. \quad (2)$$

Explain how formula (2) is obtained from (1).

40. Use formula (2) to find, to two decimal places, the volume of a cone having a diameter of 11 in. and height of 7 in.; diameter of 15 cm. and height of 2.4 dm.

102. Addition and Subtraction.—Fractions having the same denominator are added or subtracted by adding or subtracting the numerators and placing the result over the denominator. Fractions with different denominators must first be changed to equivalent fractions with a common denominator. It is best to make this common denominator as small as possible. Thus,

$$\frac{1}{6}+\frac{1}{3}=\frac{1}{6}+\frac{2}{6} \quad \text{How?} \qquad \frac{2a}{3b}-\frac{3c}{2d}=\frac{4ad}{6bd}-\frac{9bc}{6bd} \quad \text{How?}$$

$$=\frac{3}{6}=\frac{1}{2}. \quad \text{How?} \qquad =\frac{4ad-9bc}{6bd}. \quad \text{How?}$$

To check, replace a, b, c, and d each by a numerical value, say 2. Then,

$$\frac{2a}{3b}-\frac{3c}{2d}=\frac{4}{6}-\frac{6}{4}=\frac{-5}{6}. \quad \text{How? Similarly,}$$

$$\frac{4ad-9bc}{6bd}=\frac{16-36}{24}=\frac{-20}{24}=?$$ What is the conclusion?

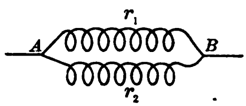
EXERCISES

Carry out the following operations and check the exercises containing literal fractions:

1. $\frac{1}{10}+\frac{2}{5}$
2. $\frac{7}{8}-\frac{5}{6}$
3. $\frac{5a}{3}+\frac{2b}{6}$
4. $\frac{3ab}{5c}+\frac{ab}{3c}$
5. $\frac{4}{ab}-\frac{5}{bc}$
6. $\frac{2a}{3c}-\frac{c}{6a}$
7. $\frac{1}{7}+\frac{3}{14}-\frac{1}{2}$
8. $\frac{2}{3}-\frac{5}{6}+\frac{1}{2}$
9. $\frac{a+b}{5}+\frac{a-b}{10}$
10. $\frac{a}{3x}+\frac{b}{6x^2}$
11. $\frac{7}{ab^2c}+\frac{5}{a^2bc}$
12. $\frac{7}{6mn^3}+\frac{8}{3mn^2}$

13. A common formula for finding electrical resistance in wires is

$$\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2}.$$



Add the last two fractions to obtain a simpler formula.

14. Find $\frac{1}{R}$ and hence R from the above formula, and also from the one you have obtained, when $r_1 = 4$ and $r_2 = 12$; when $r_1 = 7$ and $r_2 = 9$. Which formula is the simpler?

15. In changing readings of the Fahrenheit to the Centigrade thermometer it is found that

$$C = \frac{5F}{9} - \frac{160°}{9}.$$

Add the two fractions and factor the resulting numerator.

16. What is the reading on the Centigrade thermometer when that on the Fahrenheit is 18°? – 36°? 95°?

17. In a well-lighted schoolroom the area of all the windows should equal at least the area of the floor divided by 5. Express this as a formula, using a fraction.

18. What should be the area of the windows of a schoolroom that is 18′ by 22′? that is 15′ by 19′?

19. Measure your schoolroom and its window space and apply this test.

20. If there are n pupils in a schoolroom, express by a formula the number of square feet of floor space per pupil.

21. Find the floor space per pupil in a schoolroom 20′ by 28′ that seats 20 pupils; 18′ by 24′ that seats 15 pupils.

22. Find the floor space per pupil for your room.

103. Finding Common Denominators.—Common denominators can always be found from the prime factors of the denominators. The common denominator will contain each factor of the denominators the greatest number of times it occurs in any denominator. Thus,

$$\frac{8}{75}+\frac{7}{135}=\frac{8}{3\times5\times5}+\frac{7}{3\times3\times3\times5}$$

$$=\frac{72}{3\times3\times3\times5\times5}+\frac{35}{3\times3\times3\times5\times5}\qquad\text{How?}$$

$$=\frac{107}{675}.\quad\text{How?}$$

Similarly with literal fractions:

$$\frac{2}{a^2-a-2}+\frac{3}{a^2+a-6}=\frac{2}{(a-2)(a+1)}+\frac{3}{(a+3)(a-2)}$$

$$=\frac{2(a+3)}{(a-2)(a+1)(a+3)}+\frac{3(a+1)}{(a-2)(a+1)(a+3)}$$

$$=\frac{2a+6+3a+3}{(a-2)(a+1)(a+3)}=?$$

EXERCISES

Carry out the following operations and check:

1. $\frac{5}{84}+\frac{7}{60}$
2. $\frac{a}{3}+\frac{3a}{4}-\frac{5a}{12}$
3. $\frac{3}{4k^2}+\frac{7}{15k^3}-\frac{2}{k}$
4. $\frac{5}{m+2}+\frac{7}{m^2-4}$
5. $\frac{3}{77}+\frac{5}{63}$
6. $\frac{2}{a^2+2ab+b^2}+\frac{3}{a^2-b^2}$
7. $\frac{3}{x+2}-\frac{x+1}{x+3}+\frac{4}{x^2+5x+6}$
8. $\frac{2}{3(a-b)}-\frac{3}{a^2-b^2}+\frac{4}{6(a+b)}$

9. $\dfrac{4}{a^2 - 6a + 9} + \dfrac{7}{a^2 - 9}$

10. $\dfrac{5}{a^2 - 9b^2} + \dfrac{8}{a^2 - 5ab + 6b^2}$

11. $\dfrac{5}{a^2 - 4} - \dfrac{7}{a^2 - 5a + 6}$

12. $\dfrac{4}{m^2 - n^2} - \dfrac{m - 3n}{m^3 - n^3}$

13. $\dfrac{7}{f^2 + 2f - 3} + \dfrac{-5}{f^2 + f - 6}$

14. $\dfrac{-4}{y^2 - 3y - 28} + \dfrac{3}{y^2 + 9y + 20}$

15. $\dfrac{2}{a^2 - 4b^2} - \dfrac{3}{a^2 - 5ab + 6b^2}$

16. $\dfrac{3w}{2w^2 - 4w - 6} - \dfrac{2w}{w^2 - 5w + 6}$

17. $\dfrac{g + 2}{g^2 + g - 6} + \dfrac{g - 5}{g^2 - 2g - 15}$

18. $\dfrac{3}{10s^2 + 3s - 1} + \dfrac{5}{6s^2 + 7s + 2}$

19. The time, t, it takes to go a distance, d, at the velocity, v, is $t = \dfrac{d}{v}$. For a different distance and rate the time is $T = \dfrac{D}{V}$. Find the time it takes to go the two distances; that is, carry out the addition,

$$t + T = \frac{d}{v} + \frac{D}{V}.$$

20. How long will it take to go 25 mi. by freight at 15 mi. per hour and 60 mi. by auto at 30 mi. per hour ?

21. Suppose that the velocities were both the same, what would be the formula ?

22. What would be the formula for going a certain distance and returning at a different rate ?

23. Suggest a trip like the above that you would like to take, in which you use different distances and rates.

104. Mixed Numbers.—A whole number and a fraction taken together are called a **mixed number.** In numerical mixed numbers the fraction is written right after the whole number. Thus, $5\frac{2}{3}$. A literal mixed number is written with the sign plus or minus between the fraction and whole number. Thus,

$$a - \frac{c}{d} \text{ or } m + \frac{5r}{6s}.$$

105. Proper and Improper Fractions.—A fraction whose numerator can be divided by its denominator, giving a whole or mixed number, is an **improper fraction.** Such are $\frac{6}{5}$; $\frac{7}{7}$; $\frac{10a^3}{2a}$; $\frac{d-6}{d+2}$. Other fractions are called **proper** fractions. Such are $\frac{3}{5}$; $\frac{3q}{5v+7}$.

106. Reducing Mixed Numbers to Improper Fractions.—This is really adding a whole number and a fraction by reducing the whole number to a fraction of the same denominator. Thus,

$$7\tfrac{2}{3} = \tfrac{21}{3} + \tfrac{2}{3} = \tfrac{23}{3}. \qquad \text{How?}$$

$$4mn + \frac{3ah}{5m^3} = \frac{20m^4n}{5m^3} + \frac{3ah}{5m^3} = ?$$

$$2a - \frac{a^2 - 3a + 3}{a - 2} = \frac{2a^2 - 4a}{a - 2} - \frac{a^2 - 3a + 3}{a - 2} = ?$$

EXERCISES

1. Write 3 numerical and 3 literal mixed numbers.
2. Write 3 numerical and 3 literal proper fractions.
3. Write 3 numerical and 3 literal improper fractions.

Reduce the following to improper fractions and check the literal work:

4. $3\frac{2}{5}$ 5. $7\frac{3}{5}$ 6. $17\frac{3}{8}$

7. $2a + \frac{2}{3}$ 8. $5 + \frac{2w}{3z}$ 9. $d + 4e + \frac{7c}{5q}$

10. $3m + \frac{m^3}{m-3}$

11. $m - 3 + \frac{m^2 + m + 1}{m - 2}$

12. $c + b - \frac{4ab}{a - b}$

13. $g + h - \frac{2(g^2 - h^2)}{g - h}$

14. $s + r + \frac{r^3}{r^2 - sr + s^2}$

15. $t - 2u + \frac{8tu}{t + 2u}$

16. A number equals one and two-thirds times another number. Show that this is expressed by

$$N = n + \tfrac{2}{3}\,n.$$

17. Combine the two numbers to the right of the equality sign into an improper fraction.

18. The equation for amount in interest on money loaned is

$$A = P + \frac{PRT}{100}.$$

Explain this equation fully. Combine the parts to the right of the equality sign into an improper fraction.

19. Find the amount of \$ 2450 loaned for 3 yr. at 5 %.

20. Find the amount of \$ 1725 loaned for 2 yr. 3 mo. at 4.5 %.

21. Find the amount of \$ 950 loaned for 1 yr. 4 mo. at $4\frac{1}{2}$ %.

22. The length in feet of a coil of rope or belt is found as follows: Count the number of turns; multiply by 0.1309; and divide by the diameter of the coil plus the diameter of the hole at the centre. Express by a formula.

23. Compose two numerical problems for the class to solve by the use of your formula.

107. Reducing Improper Fractions to Whole or Mixed Numbers.—Literal as well as numerical improper fractions can be reduced to whole or mixed numbers. Thus,

$$\tfrac{11}{5} = \tfrac{10}{5} + \tfrac{1}{5} = 2\tfrac{1}{5}. \quad \text{How?}$$

$$\frac{5a^3b^3 + 9a}{a^2b} = \frac{5a^3b^3}{a^2b} + \frac{9a}{a^2b} = 5ab^2 + \frac{9a}{a^2b} \quad \text{How?}$$

$$= 5ab^2 + \frac{9}{ab}. \quad \text{Why?}$$

$$\frac{y+6}{y+3} = 1 + \frac{3}{y+3}. \quad \text{How?}$$

EXERCISES

Reduce the following to whole or mixed numbers:

1. $\dfrac{40}{5}$
2. $\dfrac{5x^3t^2}{x^2t^2}$
3. $\dfrac{9a^3c^4}{6a^2c^2}$
4. $\dfrac{4j^3 + 7j^5}{j^2}$
5. $\dfrac{h^4 - 3h^2 + 7h}{h}$
6. $\dfrac{354}{7}$
7. $\dfrac{m^2 - n^2}{m - n}$
8. $\dfrac{a^3 - g^3}{a - g}$
9. $\dfrac{c+5}{c+3}$
10. $\dfrac{r^3 + 27}{r+3}$
11. $\dfrac{3205}{17}$
12. $\dfrac{a^3 + b^3}{a^2 - ab + b^2}$
13. $\dfrac{2w^3 + 4w - 6}{2w}$
14. $\dfrac{s^4 - q^4}{s^2 - q^2}$
15. $\dfrac{z^4 + 2z^3 + 3z^2 + 7}{z^2 + 2z + 1}$

16. The amount to which money at interest will increase is given by the equation,

$$A = \frac{100p + prt}{100}.$$

Change this to a mixed number.

17. Gain per cent in buying and selling is

$$g\,\% = \frac{s-b}{b}.$$

Change this to a mixed number.

108. Multiplication.—Four times 2 apples is 8 apples and 4 times $\frac{2}{7}$ is $\frac{8}{7}$. The numerator is multiplied by the whole number. Similarly, $\frac{3ab^2}{2h} \times 5ac = \frac{15a^2b^2c}{2h}$. "Of" is used to indicate multiplication, just as it has been used before, so that $\frac{2}{3}$ of $\frac{6}{7} = \frac{2}{3} \times \frac{6}{7}$. In practice all multiplications are first only indicated and that expression simplified as far as possible by cancellation. Thus,

$$\frac{\cancel{4}}{\cancel{7}} \times \frac{\overset{7}{\cancel{21}}}{\cancel{5}} \times \frac{\overset{\overset{2}{\cancel{10}}}{\cancel{70}}}{\underset{\underset{\underset{4}{\cancel{12}}}{\cancel{24}}}{\cancel{96}}} = \frac{7}{4} = ? \qquad \frac{\overset{2}{\cancel{6}}p^2q}{\cancel{5}w^2r} \times \frac{\overset{2\ w^3}{\cancel{10}qw^5}}{\underset{3\,p}{\cancel{9}p^3}} = \frac{4q^2w^3}{3rp}$$

$$\frac{x^2 + 4x + 4}{v^2 + 7v + 10} \times \frac{v^2 - v - 30}{x^2 - x - 6} =$$

$$\frac{(x + 2)^{\cancel{2}}}{\cancel{(v + 5)}(v + 2)} \times \frac{(v - 6)\cancel{(v + 5)}}{(x - 3)\cancel{(x + 2)}} = \frac{(x + 2)(v - 6)}{(v + 2)(x - 3)}. \quad \text{How?}$$

EXERCISES

Multiply each of the fractions in a line by the whole number to the left. Use pencil and paper only when absolutely necessary.

1. $18 \qquad \frac{1}{2};\ \frac{3}{2};\ \frac{a}{9};\ \frac{5b}{2};\ \frac{7c}{6};\ \frac{4d^2}{3};\ \frac{11e}{18}.$

2. $36 \qquad \frac{5}{18};\ \frac{7}{12};\ \frac{2w}{6};\ \frac{a - b}{36};\ \frac{2m + n}{9};\ \frac{3k - 6}{27}.$

3. $5a \qquad \frac{3}{2};\ \frac{5}{a^2};\ \frac{7}{8};\ \frac{9c}{a^3};\ \frac{4b}{5};\ \frac{6}{5a};\ \frac{g - h}{5a};\ \frac{m - n}{10a^3}.$

4. $3v^2 \qquad \frac{2}{v};\ \frac{5}{v^2};\ \frac{7v}{3};\ \frac{5s}{3v};\ \frac{3g}{v};\ \frac{v^2}{6};\ \frac{h^3 - j}{6v^3}.$

5. $a + b \qquad \frac{1}{a + b};\ \frac{12}{a^2 - b^2};\ \frac{a - b}{a + b};\ \frac{7}{a^3 + b^3};\ \frac{a - b}{(a + b)^2}.$

Carry out the following multiplications and check as for addition and subtraction:

6. $\frac{2h}{9f^5} \times \frac{7f^3}{9h^2}$

7. $\frac{5s}{y} \times \frac{x}{10}$

8. $\frac{3m}{4} \times \frac{8j}{9m^3}$

9. $\frac{ab}{xy} \times \frac{ax}{by}$

10. $\frac{a^2b^2}{c^2d^2} \times \frac{3cd}{ab}$

11. $6m^3n \times \frac{5gh}{12mn^2}$

12. $\frac{a}{b} \times \frac{b}{c} \times \frac{c}{d}$

13. $\frac{ab}{c} \times \frac{c^2}{a^2} \times \frac{bd}{cd}$

14. $\frac{a+b}{6} \times \frac{3}{a^2-b^2}$

15. $\frac{x^2 - y^2}{5} \times \frac{10}{x - y}$

16. $\frac{1}{x^2 - 9} \times \frac{x + 3}{7}$

17. $\frac{4x}{a + b} \times \frac{(a + b)^2}{8}$

18. $(a + 2) \times \frac{5}{(a + b)^2}$

19. $(a^2 - b^2) \times \frac{-11}{a - b}$

20. $(c^3 - d^3) \times \frac{11}{c^2 + cd + d^2}$

21. $(h^3 - k^3) \times \frac{17}{h - k}$

22. $\frac{m^2+n^2}{m^3-n^3} \times \frac{m^2+mn+n^2}{m^4 - n^4}$

23. $\frac{r^2 - r - 42}{r^2 + r - 30} \times \frac{r - 5}{r - 7}$

24. $\frac{f^2 + g^2}{f + g} \times \frac{(f + g)^2}{f^4 - g^4}$

25. $\frac{m^3 + n^3}{m - n} \times \frac{m^3 - n^3}{m + n}$

26. $\left(\frac{1}{a} - \frac{1}{b}\right) \times \frac{a^2b^2}{a^2 - b^2}$

27. $\left(\frac{5}{8}\right) \times \left(-\frac{7R^2}{15M}\right) \times \left(-\frac{4M^2}{9R^3}\right)$

28. $\frac{x^4 - y^4}{(x^2 + y^2)^3} \times \frac{(x^2 + y^2)^2}{x^2 - y^2}$

29. $\left(3 + \frac{1}{a}\right) \times \left(3 - \frac{1}{a}\right)$

30. $\frac{(w + v)^2}{q - r} \times \frac{(q - r)^3}{w + v}$

31. $\left(a + \frac{b}{c}\right) \times \left(a - \frac{b}{c}\right)$

109. Multiplication of Mixed Numbers.—In multiplying small mixed numbers together first reduce them to improper fractions. With large mixed numbers use the form here given.

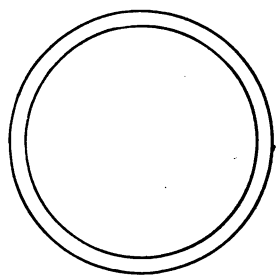
$$
\begin{array}{rl}
28\frac{2}{3} & \\
\underline{6\frac{3}{5}} & \\
\frac{2}{5} & \textit{which is } \frac{8}{5} \times \frac{2}{3} \\
16\frac{4}{5} & \textit{which is } \frac{3}{5} \times 28 \\
4 & \textit{which is } 6 \times \frac{2}{3} \\
\underline{168} & \textit{which is } 6 \times 28 \\
188\frac{6}{5} & = 189\frac{1}{5}
\end{array}
$$

EXERCISES

Carry out the following multiplications:

1.	$1\frac{1}{2} \times 1\frac{1}{2}$	**6.**	$29\frac{3}{5} \times 48\frac{5}{7}$	**11.**	$98\frac{1}{6} \times 15\frac{1}{2}$
2.	$5\frac{1}{2} \times 5\frac{1}{2}$	**7.**	$46\frac{1}{3} \times 21\frac{3}{5}$	**12.**	$98\frac{5}{6} \times 16\frac{4}{5}$
3.	$23\frac{3}{4} \times 16\frac{3}{5}$	**8.**	$65\frac{1}{9} \times 24\frac{3}{5}$	**13.**	$95\frac{5}{8} \times 45\frac{1}{2}$
4.	$8\frac{2}{3} \times 24\frac{1}{3}$	**9.**	$55\frac{1}{12} \times 48\frac{3}{4}$	**14.**	$4\frac{1}{2} \times 3\frac{1}{3} \times 2$
5.	$7\frac{1}{5} \times 52\frac{3}{4}$	**10.**	$3\frac{1}{2} \times 2\frac{1}{2} \times 6$	**15.**	$3\frac{1}{3} \times 3\frac{1}{3} \times 3\frac{1}{3}$

16. The inside diameter of a circular running-track is 140 yd. Using $3\frac{1}{7}$ for π, find the inside length of the track. If the width of the track is 5 yd., what is the outside length of the track?



17. Mary worked $4\frac{1}{3}$ wk. during vacation at \$ $3\frac{1}{4}$ per week. How much did she earn during this time?

18. What is the price of $7\frac{1}{4}$ M of lumber at \$ $45\frac{1}{2}$ per M? What does M mean?

19. Find the cost of $6\frac{1}{4}$ bu. apples at \$ $2\frac{1}{5}$ per bushel; at \$ $3\frac{1}{4}$ per bushel.

Hold a number contest on multiplying fractions.

110. Fractional Equations.—Equations containing fractions may be changed to other true equations without fractions by multiplying both members by a number containing each denominator as a factor. The two results will be equal. Why? Each member will be without fractions. Why? Thus,

$\frac{a}{3} + \frac{1}{6} = \frac{a+1}{2}$	(1)	
$2a + 1 = 3a + 3$	(2)	Multiply (1) by 6.
$2a - 3a = 3 - 1$	(3)	How?
$-a = 2$	(4)	How?
$a = -2.$	(5)	Check in (1).

EXERCISES

Solve the following equations and check:

1. $\frac{t}{2} + 3 = \frac{t-2}{3}$
2. $\frac{x}{2} + \frac{x}{3} = \frac{375}{6}$
3. $\frac{x+5}{2} - \frac{x+1}{4} = 3$
4. $\frac{y-7}{5} + 2 = \frac{y+8}{10}$
5. $\frac{17}{2a} = \frac{31}{2} - \frac{7}{a}$
6. $\frac{6}{a} - \frac{1}{a} = \frac{5}{3}$
7. $\frac{c-2}{4} - \frac{c-4}{6} = \frac{2}{3}$
8. $\frac{5}{4m} - \frac{2}{3m} = \frac{7}{12}$
9. $\frac{5x+4}{5} - \frac{8x-9}{10} = \frac{51}{6}$
10. $5x - \frac{x+2}{2} = 71$
11. $a - \frac{3-a}{3} = \frac{17}{3}$
12. $\frac{11+m}{6} - \frac{10-m}{3} = 1$
13. $\frac{5t+4}{2t} - \frac{11t-2}{6t} = 3$
14. $k + \frac{1}{k} = 2$
15. $\frac{g^2}{3} - \frac{4}{3} = 4$
16. $x - 1 = \frac{12}{x}$

17. $\frac{3}{4x} + \frac{9}{x} = \frac{31}{12} + \frac{2}{x}$

18. $\frac{m}{2} + \frac{m}{3} = -\frac{35}{6}$

19. $\frac{k}{5} - \frac{2k}{3} = -\frac{28}{15}$

20. $\frac{5t+4}{2t} - \frac{11t-2}{6t} = \frac{17}{15}$

21. $\frac{3m-5}{2} - \frac{7m-13}{6} = -\frac{2}{3}$

22. $\frac{8f-4}{5} - \frac{3f-2}{8} = \frac{3f}{2}$

23. Express mathematically: one-fifth of a certain number; the square of a number divided by 7; the sum of one-sixth and one-eighth of a number.

24. The sum of a fifth and a ninth of a number is 14. What is the number?

25. The sum of one-half and three-fourths of a number equals the number plus 2. Find the number.

26. The difference between two numbers is 18. If the larger number is represented by N, what is the smaller number?

27. If the smaller of the two numbers of Ex. 26 is divided by the larger, the quotient will be $\frac{1}{2}$. Find the two numbers.

28. The square of a certain number divided by 3 equals 27. Find the number.

29. One-fourth of the square of a certain number and one-eighteenth of the square of the number equals 11. Find the number.

30. Mabel was asked how old she was. Her reply was, "One-third of my age plus one-fourth of my age is 7 yr." How old was she?

31. Dividend less remainder divided by quotient equals divisor. Express this as an equation. State by an equation what the remainder equals in terms of the other numbers.

111. Division.—Division is the reverse of multiplication. We ask what number multiplied by the divisor gives the dividend. To divide $\frac{4}{5}$ by $\frac{2}{3}$ is to find the fraction that multiplied by $\frac{2}{3}$ gives $\frac{4}{5}$. This fraction must have 4 a factor of the numerator and 5 a factor of the denominator. It must also have 3 a factor of the numerator and 2 a factor of the denominator, so that when multiplied by $\frac{2}{3}$ the 2's and 3's cancel.

$$\frac{2}{3} \times ? = \frac{4}{5}, \text{ or } \frac{4}{5} \div \frac{2}{3} = \frac{4}{5} \times \frac{3}{2} = ?$$

Similarly,

$$\frac{6rd^3}{5j^2k} \div \frac{9f^2d}{10jr} = \frac{6rd^3}{5j^2k} \times \frac{10jr}{9f^2d} = ?$$

Hence, to divide by a fraction, invert it and multiply the dividend by the fraction inverted.

To divide a fraction by another fraction having the same denominator, merely divide the numerators. Thus,

$$\frac{15}{7} \div \frac{5}{7} = 15 \div 5 = ? \quad \frac{12}{7} \div \frac{5}{7} = \frac{12}{5} = ?$$

EXERCISES

Carry out the following operations and check:

1. $\frac{18}{7} \div \frac{6}{7}$
2. $\frac{3}{4} \div \frac{8}{9}$
3. $\frac{9}{14} \div \frac{6}{35}$
4. $\dfrac{15a^2}{7} \div \dfrac{5a}{7}$
5. $\dfrac{12w^3}{35g^2} \div \dfrac{8w^2}{7g^4}$
6. $\frac{7}{8} \div \frac{35}{12}$
7. $\frac{36}{35} \div \frac{27}{14}$
8. $27\frac{4}{5} \div \frac{3}{5}$
9. $\dfrac{ab}{x^3} \div \dfrac{a^2b^2}{xy^2z}$
10. $\dfrac{42q^5}{75r^3} \div \dfrac{36q^3}{25r^5}$
11. $24 \div \frac{2}{3}$
12. $36\frac{3}{4} \div \frac{2}{3}$
13. $24\frac{3}{5} \div \frac{3}{10}$
14. $34s^2t^3 \div \dfrac{st^2}{5}$
15. $\dfrac{28m^7n^3}{45g^3h} \div \dfrac{14m^3n^2}{27gh^3}$
16. $\frac{3}{4} \times \frac{5}{9} \div \frac{2}{3}$
17. $\frac{7}{9} \div \frac{8}{15} \times \frac{12}{35}$
18. $\dfrac{1}{a+b} \div \dfrac{3}{a^2-b^2}$
19. $\frac{5}{12} \times \frac{6}{7} \div \frac{8}{15}$
20. $\frac{6}{5} \div \frac{8}{15} \div \frac{9}{20}$
21. $\dfrac{k+2}{k^2-6k+9} \div \dfrac{k^2+5k+6}{2k-6}$

22. $\dfrac{a^2-b^2}{m+n} \div \dfrac{a+b}{m^3+n^3}$

23. $\dfrac{k^2-9}{m+2} \div \dfrac{k-3}{m^2-4}$

24. $\dfrac{r^3+t^3}{r-t} \div \dfrac{r+t}{r^3-t^3}$.

25. $\dfrac{c^3-d^3}{h+1} \div \dfrac{c^2-d^2}{h^2-1}$

26. $3(r-u)^2 \div \dfrac{15(r-u)}{7}$

27. $\dfrac{m^2-n^2}{5h^2} \div \dfrac{m^2-n^2}{20h^3} \times \dfrac{m+n}{12}$

28. $\dfrac{w^2+6w+9}{y^2-6y+5} \div \dfrac{w+3}{y-5}$

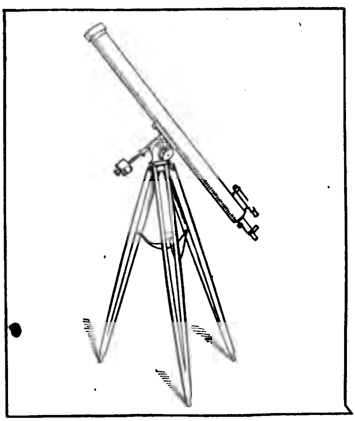
29. $\dfrac{t^2+4t+4}{t^2-8t+15} \div \dfrac{3t^2+4t-4}{t^2-9}$

30. $\dfrac{a+b}{a-b} \times \dfrac{a^2-2ab+b^2}{a^2+2ab+b^2} \div \dfrac{a^3-b^3}{2}$

31. $\left(b-\dfrac{1}{b}\right) \times \dfrac{b^2}{b+1} \div \dfrac{5b}{4}$

32. $\left(1-\dfrac{4}{g^2}\right) \div \dfrac{(g+2)^2}{5g^3} \times \dfrac{3(g+2)}{20(g-2)}$

33. Astronomers use a formula found by dividing $\dfrac{4\pi^2r^2}{Gp}$ by $\dfrac{4\pi^2R^2}{GP}$. Carry out this division.



34. How long will it take Nina, earning \$ $2\frac{1}{4}$ per week, to earn \$ $8\frac{3}{4}$?

35. A cubic foot of water weighs nearly $62\frac{1}{2}$ lb. If there are $7\frac{1}{2}$ gal. in a cubic foot, how much does a gallon of water weigh ?

36. How many bushels of apples can be bought for \$ $5\frac{1}{2}$ at \$ $2\frac{1}{5}$ per bushel ?

112. Division by Mixed Numbers.—In order to divide by an integer or a mixed number, first reduce the divisor to an improper fraction. Thus,

$$24\tfrac{3}{4} \div 2\tfrac{1}{9} = 24\tfrac{3}{4} \div \tfrac{19}{9} = 24\tfrac{3}{4} \times \tfrac{9}{19} = ?$$
$$8\tfrac{2}{3} \div 5 = 8\tfrac{2}{3} \times \tfrac{1}{5} = ?$$

Division of a mixed number by a whole number may often be shortened by dividing the whole number by the divisor first, then adding any remainder to the fraction in the dividend, and finally dividing this fraction by the divisor. Thus,

$$45\tfrac{2}{3} \div 4 = 11 + (1 + \tfrac{2}{3}) \div 4 = 11 + \tfrac{5}{3} \div 4 = 11\tfrac{5}{12}.$$

EXERCISES

Carry out the following operations:

1. $12 \div 1\tfrac{1}{3}$	5. $6\tfrac{1}{7} \div 3\tfrac{2}{5}$	9. $15\tfrac{2}{3} \div 4$
2. $36 \div 2\tfrac{2}{5}$	6. $4\tfrac{1}{3} \div 3\tfrac{1}{9}$	10. $36\tfrac{3}{8} \div 8\tfrac{1}{4}$
3. $2\tfrac{1}{4} \div 1\tfrac{7}{8}$	7. $4\tfrac{7}{8} \div 3\tfrac{1}{4}$	11. $3\tfrac{1}{2} \div 37\tfrac{3}{8}$
4. $5\tfrac{1}{3} \div 2\tfrac{1}{2}$	8. $42 \div 5\tfrac{2}{7}$	12. $44\tfrac{1}{4} \div 7$

13. $4a \div \left(a - \dfrac{a}{2}\right)$

14. $\dfrac{4b}{3} \div \left(b - \dfrac{b}{3}\right)$

15. $15 \div (2x - 3)$

16. $\left(a - \dfrac{1}{a}\right) \div \left(a + \dfrac{1}{a}\right)$

17. $\left(3 + \dfrac{2}{a}\right) \div \left(9 - \dfrac{4}{a^2}\right)$

18. $\left(1 + \dfrac{2b}{a}\right) \div \left(a - \dfrac{4b^2}{a}\right)$

19. Using π as $3\tfrac{1}{7}$, find the diameter in yards of a circular running-track that is $\tfrac{1}{4}$ mi. long; $\tfrac{1}{8}$ mi. long.

20. How many bushels of apples can be bought for \$ 5 at \$ $1\tfrac{3}{4}$ per bushel? for \$ 6 at \$ $2\tfrac{1}{5}$ per bushel?

21. How many weeks must Janie work during her vacation to earn \$ 10 at \$ $2\tfrac{3}{4}$ per week? \$ $12\tfrac{1}{2}$ at \$ $3\tfrac{1}{4}$ per week?

22. A bushel of apples or potatoes measures $1\tfrac{1}{2}$ cu. ft. How many bushels will a bin hold that is $4' \times 6' \times 9'$?

23. Measure some wagon-box to see how many bushels of apples or potatoes it will hold.

113. Equations with Fractional Roots.—Many equations have fractional roots. Thus,

$$\frac{3k}{2} - \frac{6k+3}{5} = 4 \qquad (1)$$

$$15k - 12k - 6 = 40 \qquad (2) \text{ How?}$$

$$15k - 12k = 40 + 6 \qquad (3) \text{ How?}$$

$$3k = 46 \qquad (4)$$

$$k = 15\tfrac{1}{3}. \qquad (5) \text{ Check in (1)}$$

EXERCISES

Solve and check the following:

1. $5e = 7 + 3e$
2. $8t - 4 = 3t + 5$
3. $3(m - 7) = 5(m + 4)$
4. $-7(a - 4) - 2a = 0$
5. $\frac{t}{2} + 5t + \frac{1}{4} = 3$
6. $\frac{m+2}{3} - m = \frac{1}{2}$
7. $\frac{5m+2}{5} - 2m = -\frac{1}{5}$
8. $\frac{4r+3}{2} - 4 = r$
9. $\frac{x}{2} - \frac{x+1}{3} = \frac{1}{6}$
10. $\frac{2a-5}{3} + \frac{4}{9} = -1$
11. $2x^2 - 5x + 3 = 0$
12. $3y^2 - 10y + 7 = 0$
13. $(8a - 1)^2 = (2a + 3)^2$
14. $(7c - 3)^2 = (2c - 1)^2$
15. $2(3x - 5) - 3(x - 7) = 0$
16. $(3x - 2)(3x + 2) = 9x^2 + 8x$
17. $(3m + 7)(3m + 2) = 9m^2 + 23$
18. $\frac{3r+2}{5} - \frac{5r-2}{6} = \frac{1}{2}$
19. $\frac{2}{3} - \frac{4L+7}{5} = -\frac{7}{12} - L$
20. $x + 8 = -\frac{7}{x}$
21. $\frac{a}{4} - \frac{1}{2} + \frac{1}{4a} = 0$
22. $\frac{2a-7}{9} - \frac{3}{5}(a + 5) = \frac{51}{10}$
23. $4s^2 - 9s = -2$
24. $3g^2 + 16g = 3(g^2 - 4)$

25. If l, w, and h are given in inches, the number of gallons in a rectangular tank is

$$G = 0.004329lwh.$$

To what depth must a tank 30″ by 18″ be filled to contain 30 gal.? to contain 12 gal.?

26. If you have a rectangular tank in your school, find to what depth it must be filled to contain 5 gal.; 11 gal.

27. Find the value of F if C is 15° in the equation

$$C = \frac{5F}{9} - \frac{160°}{9}.$$

28. Engine-builders use the following formula in making safety-valves,

$$A = \frac{22.5G}{P + 8.62}.$$

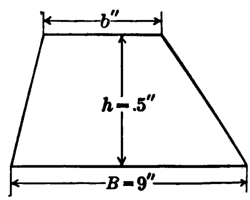
If A is 8 and G is 40, what is the value of P?

29. Write the formula for the average of a basket-ball team from the number of games won and lost. If the average of a team last year was 0.583, how many games would the team expect to lose for every 5 won? What does a fraction mean that is less than 0.5? that is more than 0.5?

30. How many would the team expect to win for every 6 games lost? for every 9 games lost?

31. If you have a basket-ball team, compute its average. How many games do you expect to win out of 8 played? of 11 played?

32. Write the formula for the area of a trapezoid. Use the dimensions of this figure and find b when the area is 20 sq. in.; 26 sq. in.; 36 sq. in.



33. The bases of a trapezoid are 23 dm. and 18 dm. respectively. Find the altitude if the area is 110 dm.^{2}; if the area is 150 dm.^{2}; if the area is 320 dm.^{2}.

114. Literal Roots.—An equation containing two or more literal numbers may be solved for any one of the literal numbers. Its root will contain all of the other literal numbers. Thus,

$$3a + 2t = 7 \qquad (1)$$

$$3a = 7 - 2t \qquad (2) \quad \text{How?}$$

$$a = \frac{7 - 2t}{3}. \qquad (3) \quad \text{Check in (1)}$$

EXERCISES

Solve for k in Exs. 1–20. Check all roots:

1. $5k + 3 = at$
2. $5 - 4k = 2b$
3. $2kt + 3 = 4t$
4. $7ka - 3a = 9$
5. $3k + 7b = 4t$
6. $2kt^2 = t - 1$
7. $Ak = A^2$
8. $(a + b)k = (a + b)^2$
9. $(a - b)k = a^2 - b^2$
10. $ak - bk = a^2 - b^2$
11. $(x + 7)k = x^2 - 49$
12. $xk + 7k = (x + 7)^2$
13. $mk + nk = m^2 - n^2$
14. $ak - bk = 4$
15. $rk = 5 + sk$
16. $(a + b + c)k = 2(a + b + c)$
17. $ak + bk + ck = 3a + 3b + 3c$
18. $ak + bk = a + b + c - ck$
19. $5k^2 - 500t^2q^2 = 0$
20. $(7r + 2)k = 14r^2 - 17r - 6$
21. $C = 2\pi r$. Solve for r.
22. $v = \pi r^2 h$. Solve for h.
23. $V = \frac{1}{3}\pi r^2 h$. Solve for h.
24. $A = \frac{1}{2}(b + b')h$. Solve for h.
25. $9C = 5(F - 32)$. Solve for F.

It is often necessary to change a formula to another form by solving the equation for some certain letter.

26. State by a formula that the distance travelled equals the velocity times the time. Solve this for v; for t.

27. One form of a mechanic's formula is

$$S = T - \frac{1.733}{N}.$$

Solve this for T; solve for N.

28. Stand-pipes, as here shown, are often used to secure water pressure. The pressure in pounds per square inch, p, for a stand-pipe h feet high is

$$p = 0.434h.$$

Solve this for h.

29. What will be the pressure per square inch and per square foot for a stand-pipe 50′ high? 60′ high?

30. If a city has a stand-pipe 65′ high, what pressure per square inch must its water-pipes be able to stand?

31. If there is a stand-pipe in your city, find its height and from that the pressure per square inch upon the water-pipes.

32. Solve the equation in Ex. 27, page 112, for F.

33. Solve the equation in Ex. 28, page 112, for P.

34. Solve the equation in Ex. 28, page 112, for G.

35. The amount of money loaned at interest is

$$A = P + \frac{PRT}{100}.$$

Solve this for P; solve for R; solve for T.

36. What principal amounts to \$ 2500 in 2 yr. at 5 %?

37. In what time will \$ 1000 amount to \$ 1200 at 5 %?

V

COMMON FRACTIONS—DECIMALS—PER CENTS

115. Common Fractions and Decimals.—Decimals are really common fractions whose denominators are powers of 10. Hence, to change a decimal to a common fraction, merely place the required denominator under the decimal part and reduce to lowest terms. Thus,

$$0.032 = \tfrac{32}{1000} = \tfrac{4}{125}.$$

A common fraction is reduced to a decimal by carrying out the division of the numerator by the denominator. Thus,

$$\tfrac{14}{25} = 14 \div 25 = 0.56.$$

EXERCISES

Change to common fractions:

1. 0.45	**3.** 25.055	**5.** 85.075	**7.** 145.0705
2. 25.15	**4.** 0.0085	**6.** 7.0845	**8.** 39.0105

Change to three-place decimals:

1. $\frac{1}{8}$	**5.** $15\frac{5}{7}$	**9.** $13\frac{25}{45}$	**13.** $27\frac{117}{128}$
2. $\frac{15}{16}$	**6.** $81\frac{2}{9}$	**10.** $44\frac{8}{15}$	**14.** $35\frac{27}{40}$
3. $\frac{7}{8}$	**7.** $33\frac{8}{11}$	**11.** $7\frac{2}{3}$	**15.** $372\frac{15}{256}$
4. $\frac{1}{3}$	**8.** $18\frac{19}{80}$	**12.** $9\frac{3}{7}$	**16.** $173\frac{217}{256}$

17. Show that $3\frac{1}{7}$ is a close approximate value for π, which as a decimal is nearly 3.1416.

18. Show that the following formulas are equivalent, $V = \frac{22}{42}D^3$ and $V = .5238D^3$.

Hold a number contest on the reduction of decimals to common fractions and common fractions to decimals.

116. Aliquot Parts.—How do you multiply by 10, 100, etc.? How do you multiply by 20, 300, etc.? How do you divide by 10, 100, etc.? How do you divide by 30, 200, etc.? An **aliquot part** of a number is a number which will divide it and give a whole number as a quotient. Thus, 5 and 2 are aliquot parts of 10. Give two aliquot parts of 100. Aliquot parts are used to simplify multiplication and division. For instance, to multiply by 25 multiply by 100—annex two zeros—and divide by 4. Why? To divide by 25 multiply by 4 and divide by 100—point off two places. Why? The most useful aliquot parts of 100 are 2, 5, $12\frac{1}{2}$, $16\frac{2}{3}$, 25, $33\frac{1}{3}$, and 50. As $66\frac{2}{3}$ is $\frac{2}{3}$ of 100 and 75 is $\frac{3}{4}$ of 100, these are also used to some extent. Make use of these simplifications whenever possible. Thus, $324 \times 633\frac{1}{3} = 324 \times 6\frac{1}{3} \times 100$. Why? $324 \times 6\frac{1}{3} \times 100 = ?$

EXERCISES

Carry out the following operations without pencil and paper, as far as possible:

1. 532×25
2. $624 \div 25$
3. 753×50
4. $714 \times 16\frac{2}{3}$
5. $634 \div 50$
6. $912 \times 66\frac{2}{3}$
7. 425×300
8. $7456 \div 400$
9. 372×575
10. 531×400
11. $54645 \div 400$
12. $642 \times 33\frac{1}{3}$
13. $642 \div 33\frac{1}{3}$
14. 37.94×50
15. $6.18 \times 233\frac{1}{3}$
16. 416×375
17. $5034 \div 16\frac{2}{3}$
18. $64.32 \div 25$
19. 17.16×325
20. $51.48 \div 400$
21. $436 \times 533\frac{1}{3}$
22. $534 \times 316\frac{2}{3}$
23. $78054 \div 500$
24. $636 \times 566\frac{2}{3}$
25. $6474 \div 33\frac{1}{3}$
26. $4638 \div 2000$
27. $927 \times 533\frac{1}{3}$
28. $7782 \div 500$
29. 7048×25
30. $4238 \div 25$

31. Suggest 10 exercises for the class to solve in which they can make use of aliquot parts.

32. Find the length of a meter in inches from the table and compute the length of 25 m.; of 325 m.; of 74 m.

33. How many tons of coal are there in 4680 lb.? in 3260 lb.? in 7520 lb.? in 7940 lb.?

34. Find the cost of the coal in Ex. 33 at $ 6.25 per ton.

35. A merchant adds $0.33\frac{1}{3}$ to the cost of goods for expense and profit. For what does he sell goods costing $ 45? costing $ 315? costing $ 705?

36. Mr. Wilson is offered 0.75 of the cost of an auto for which he paid $ 728. How much is he offered?

117. Per Cent.—The number of hundredths in a decimal is often called per cent by the business men. Thus, 0.15 is 15 per cent; 0.234 is 23.4 per cent; 0.067 is 6.7 per cent. What is 0.34? 0.70? 0.034? 0.09? 0.03? 0.125? 0.134? Similarly, 3.25 is 325 per cent. What is 4.65? 2.15? 7.06?

The sign used is %. Hence, 34 per cent is written 34 %. Write the above per cents by using this sign.

EXERCISES

1. Change each of the following to per cents: $\frac{3}{4}$; 0.35; 3.25; $3\frac{1}{5}$; 0.45; 0.345; 2.345; 0.045; $\frac{4}{5}$; $\frac{2}{3}$; $\frac{3}{5}$; 0.0245; 0.567; $3\frac{2}{5}$; $\frac{1}{8}$; $\frac{2}{84}$; 0.341; 0.045; $1\frac{1}{2}$.

2. Change the following to decimals and common fractions: 20 %; 45 %; 64 %; 35.4 %; 4.5 %; $33\frac{1}{3}$ %; $12\frac{1}{2}$ %; 0.45 %.

3. Suggest 10 common fractions to change to per cent.

4. Suggest 10 decimals to change to per cent.

5. Suggest 10 per cents to change to common fractions and to decimals.

118. Problems Involving Per Cent.—Reread Art. 46 on the solution of problems. Whenever a per cent occurs in a problem, first change it to a decimal.

A house was sold for \$ 2400, which was a loss of 25 % of the cost. What was the cost of the house?

	Cost less .25 of cost equals \$ 2400			(1)
	C − .25 C	=	\$ 2400	(2)
hence,	.75 C	=	\$ 2400	(3)
and	C	=	\$ 2400 ÷ .75 = \$ 3200.	(4)

Check: 25 % of \$ 3200 is \$ 800; \$ 3200 − \$ 800 = \$ 2400.

If a per cent is asked for, the root of the equation solved will be a decimal or a common fraction which must be reduced to a per cent.

An automobile bought for \$ 870 is sold for \$ 580. What per cent is this of the buying price?

$$870 \times p = 580 \qquad (5)$$

$$p = 580 \div 870 = \tfrac{2}{3} \qquad (6)$$

But $\frac{2}{3}$ equals $66\frac{2}{3}$ %

hence,

$$p = 66\tfrac{2}{3}\,\%. \qquad (7)$$

EXERCISES

Express as equations the following business statements:

1. Cost plus gain equals selling price.
2. Cost plus gain per cent of cost equals selling price.
3. Cost equals selling price less gain per cent of cost.
4. Make the statements corresponding to Exs. 1, 2, and 3 for goods sold at a loss. Also from these write out the equations saying the same thing.

5. C + 10 %C = ? %C
6. C + 25 %C = ? %C
7. *C* − 25 % = ? %C
8. C − 10 % = ? %C
9. C + 3.5 %C = ? %C
10. C − 4.5 %C = ? %C

11. If 65 % of the cost of a suit of clothes is $ 26, what did the suit cost?

12. What is the cost of a suit of clothes, if 115 % of the cost of the suit is $ 34.50?

13. After using an automobile for two years, a man sold it for $ 819, which was 35 % below what he paid for it. What did the automobile cost him?

14. A house sold for $ 3600, which was 10 % below the cost. What did the house cost?

15. A horse was sold for $ 230. Find the cost if 15 % was gained by the sale.

16. A dealer sells suits so as to gain 20 %. What does he pay for suits that he sells for $ 24? for $ 36? for $ 30?

17. Henry contracts to sell papers for which he is to receive 40 % of the selling price. What per cent of the selling price did Henry pay for the papers? If he sold papers at 5 ¢ each, what did he gain on every paper sold?

Express the statements in Exs. 18 and 19 as equations:

18. Loss equals cost times loss per cent.

19. Gain equals cost times gain per cent.

20. If $ 5 is gained in selling an article which costs $ 15, what is the gain per cent?

21. If $ 2 is lost by selling an article out of season for $ 6, what is the loss per cent?

22. If $ 5 is gained in selling an article that costs $ 25, what is the gain per cent?

23. If $ 6 is lost in selling an article which costs $ 36, what is the loss per cent?

24. A bicycle bought for $ 55 was sold for $ 35. Find the loss and the loss per cent.

25. John sells papers at 5 ¢ each for which he pays 3 ¢. Find his gain per cent.

26. Why do merchants often give discounts to their customers? See how many reasons you can name.

27. State as a formula that the selling price of goods is $c\ \%$ of the marked price.

28. Explain that the formula stating that goods sold $d\ \%$ below the marked price is

$$S.\ P. = (1 - d\ \%)M.\ P.$$

29. What is the selling price of goods marked at \$ 36, if the discount is 10 %? 25 %? $33\frac{1}{3}$ %? 35 %?

30. Solve the equation in Ex. 28 for $M.\ P.$

31. What must be the marked price of goods which are to sell at \$ 48, so that a discount may be given of 25 %? of $33\frac{1}{3}$ %? of 15 %?

32. Solve the equation in Ex. 28 for $d\ \%$.

33. Find the per cent of discount if goods marked at \$ 10 are sold for \$ 8; marked at \$ 30 are sold for \$ 18.

34. The radius of a circle when increased 10 % is $1.1\ r$. What will be the radius if increased 20 %? 50 %? 40 %? 65 %?

35. What is the area of a circle whose radius is r? What is the area if r is increased 10 %? What is the increase in area? The increase in area is what per cent of the area of the circle with radius r?

36. Answer the questions in Ex. 35 when the increase in the radius is 25 %; 50 %; 100 %.

37. What is the area of the floor of a room w feet wide and l feet long? What will be the area if each dimension is increased 10 %? What is the increase in area? The increase in area is what per cent of the original area?

38. Measure your schoolroom and answer the questions in Ex. 37.

119. Complex Fractions.—Either or both terms of a fraction may themselves be fractions, or operations with fractions and whole numbers. These fractions are called **complex fractions.** Such are

$$\frac{\frac{2}{3}}{\frac{7}{8}};\quad \frac{\frac{3}{4}-\frac{2}{5}}{\frac{5}{6}+\frac{1}{4}};\quad \frac{5-2\frac{1}{3}}{\frac{7}{9}+3\frac{1}{2}};\quad \frac{\frac{2a}{3v}}{\frac{3f}{h}};\quad \frac{5fg-\frac{3e}{w}}{\frac{3j}{k}+2w^2}.$$

Since a fraction is an indicated division, a complex fraction may be simplified by merely treating it as the numerator divided by the denominator. Thus,

$$\frac{\frac{3}{4}}{\frac{5}{6}} = \frac{3}{4} \div \frac{5}{6} = ?$$

$$\frac{\frac{2a^2}{3b}+\frac{3c}{b^2}}{\frac{5ac}{6b^2}} = \left(\frac{2a^2}{3b}+\frac{3c}{b^2}\right) \div \frac{5ac}{6b^2} = \frac{2a^2b+9c}{3b^2} \div \frac{5ac}{6b^2} = ?$$

EXERCISES

Carry out the following operations and check:

1. $\dfrac{2\frac{1}{2}}{3\frac{1}{4}}$

2. $\dfrac{\frac{5}{7}\times 2\frac{1}{4}}{3\frac{1}{2}+2\frac{1}{4}}$

3. $\dfrac{\frac{a}{b}}{\frac{a}{b^2}}$

4. $\dfrac{\frac{1+n}{n}}{n^2-\frac{1}{n^2}}$

5. $\dfrac{3\frac{1}{2}+1\frac{3}{4}}{2\frac{1}{3}}$

6. $\dfrac{\frac{3}{4}\times\frac{4}{9}}{\frac{6}{5}\times\frac{10}{27}}$

7. $\dfrac{\frac{2}{a^2-b^2}}{\frac{1}{a+b}}$

8. $\dfrac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{a}-\frac{1}{b}}$

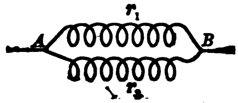
9. $\dfrac{4\frac{3}{4}-2\frac{1}{2}}{\frac{1}{2}\times 1\frac{1}{2}}$

10. $\dfrac{1-\frac{3}{20}}{1+\frac{5}{12}}$

11. $\dfrac{a+\frac{1}{a}}{a-\frac{1}{a^3}}$

12. $\dfrac{\frac{x^2-1}{x^2+1}}{\frac{1-x^4}{(1+x^2)^2}}$

13. r_1 and r_2 are the electrical resistances of each of the wires between A and B, shown in the figure. The total resistance, R, due to these two wires is



$$R = \frac{1}{\frac{1}{r_1} + \frac{1}{r_2}}.$$

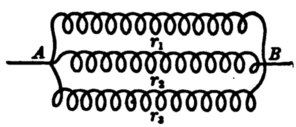
Show that this simplifies to

$$R = \frac{r_1 r_2}{r_1 + r_2}.$$

14. The unit of electrical resistance is called the **ohm.** If r_1 is 6 ohms and r_2 is 8 ohms, what is R?

15. If A and B had been connected with three wires whose resistances were r_1, r_2, and r_3 each, then,

$$R = \frac{1}{\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}}.$$



Simplify this, as in Ex. 13.

16. Find R when r_1 is 2, r_2 is 3, and r_3 is 4.

17. The velocity with which water will flow from an opening at the base of a stand-pipe is given by the formula to the right. Change the fraction from the compound to a simple fraction.

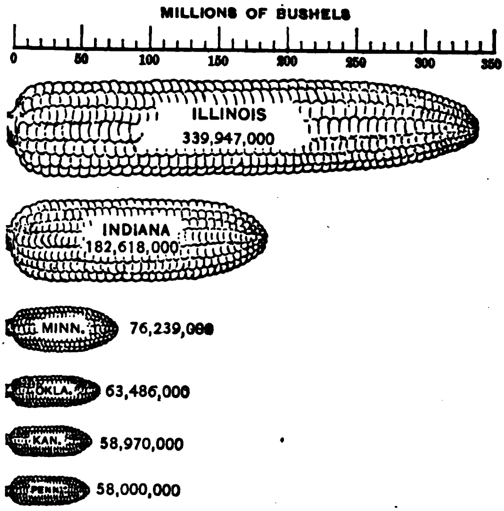
$$V^2 = \frac{H \times 2500}{l \times \frac{13.9}{d}}$$

18. Explain the interest equation, $I = \frac{PRT}{100}$. From this, $P = \frac{I}{\frac{RT}{100}}$. Change this to a simple fraction.

19. What principal will give \$ 60 in 2 yr. at 5 % interest?

VI

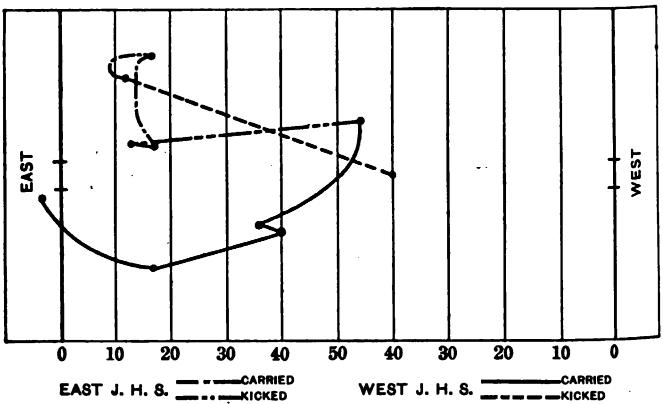
GRAPHS



120. Picture Graphs.—The above pictures give an easy comparison of the amount of corn raised in each of several states during the year 1916. Any drawing used to show comparisons in this manner is called a **graph.** Graphs like the above are called **picture graphs.** Why? Be on the lookout for graphs in books, magazines, and newspapers while studying this chapter.

Read the above numbers. Do the numbers or the pictures show the comparisons the more easily? the more exactly?

121. Football Graph.—The plays in a football game can be shown clearly by a graph.



EXERCISES

1. How many yards was the ball "punted" toward the goal of the East Junior High? toward the goal of the West Junior High?

2. What was the actual distance that the ball was "punted" in each of the above two instances?

3. The distance "punted" by East Junior High toward their opponents' goal was what per cent of the total distance punted by East Junior High?

4. Answer 3 for West Junior High.

5. How far was the ball carried toward the goal of the East Junior High? toward the goal of the West Junior High?

6. What was the total distance that the ball was carried by the West Junior High? What per cent of this distance *was toward* their opponents' goal?

7. Answer Ex. 6 for the East Junior High.

8. Express the percentages in Exs. 3 and 4 as decimals carried out to .001.

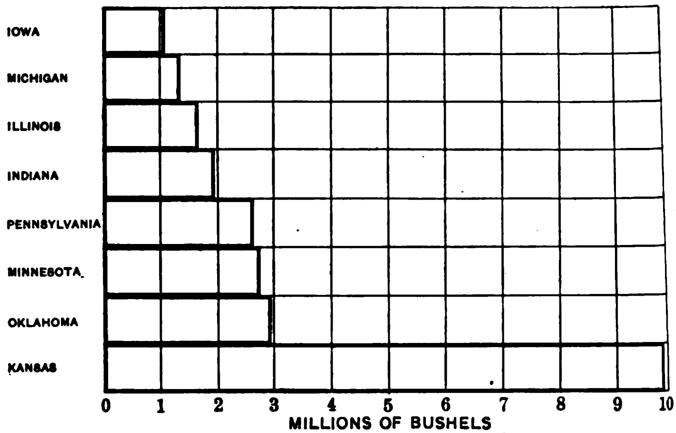
9. Express the percentages in Exs. 6 and 7 as decimals carried out to .001.

10. Draw a football field several times larger than the one on page 124. Graph the following data, taking your school and some other school that you know for the two teams:

Team *A* "kicked off" to team *B* and the ball was caught on the 20-yard line 25 yd. from the right side-line. Team *B* carried the ball directly toward *A*'s goal 15 yd. before it was "down." First down, 3 yd. forward and 10 yd. to the left. Second down, 2 yd. forward and 20 yd. to the left. Third down, 3 yd. forward and 30 yd. to the right. Fourth down, punt 35 yd. forward and 10 yd. from right side-line. The ball was brought forward by team *A* 10 yd. First down, 5 yd. forward and 20 yd. to the left. Second down, 8 yd. forward and 5 yd. to the left. First down, 15 yd. forward, and 10 yd. from right side-line. First down, thrown back 5 yd. and 25 yd. to the left. Second down, 11 yd. forward and 10 yd. to the left. Third down, 7 yd. forward and 5 yd. to the right. First down, end run across goal-line and down behind goal-line 20 yd. from the left side-line.

11. Make up a set of six plays coming one after the other for the class to graph. The class will graph the most interesting plays. Try to make yours the one to be graphed.

Hold a number contest on adding and subtracting fractions.



122. Rectangle and Circle Graphs.—Rectangles of the same width but of different lengths are often used for comparisons, as in the graph above. Such graphs are also called **bar graphs.** Graphs are made by dividing a rectangle into different-size parts for comparisons. Above is a bar graph showing the number of bushels of wheat raised in eight states during the year 1916. On the opposite page is found an illustration of a circle graph.

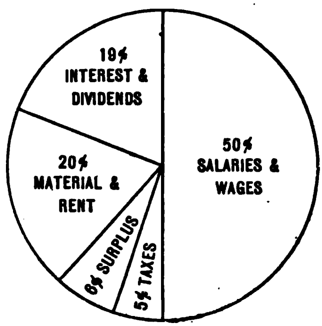
EXERCISES

1. Explain the graph above. About how many millions and fraction of a million bushels of wheat were raised in each of the states mentioned? State each as a one-place decimal. Why can you not tell exactly? Can you tell exactly enough to give a useful comparison?

2. The number of bushels of wheat raised in Pennsylvania in 1916 is what per cent of that raised in Kansas? in Minnesota? in Illinois? in Iowa?

3. Make a bar graph for the following: the average wheat yield per acre for 1898–1907 was 32.6 bu. in Great Britain; 28.4 bu. in Germany; 20.8 bu. in France; 13.9 bu. in Austria-Hungary; 9.3 bu. in Russia. Change each number to a whole number. What happens to fractions less than .5? to those .5 or more? If uncertain, see Art. 24. In the graph let $\frac{1}{4}$ in. represent 1 bu.

4. The circular graph to the right gives the use of income of the American Telegraph and Telephone Company for 1911. What is the sum of the angles at the centre of a circle? What should be the number of degrees in the angle of each sector in the graph? Measure each to see if the graph is correct.



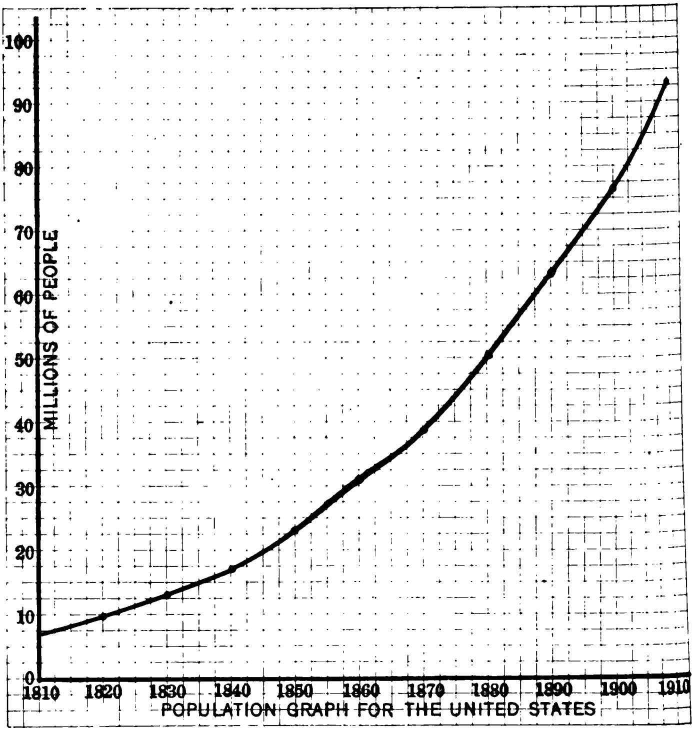
5. Make a bar or a circular graph for the following: The number of cotton bales produced in 1915 were

United States	13,000,000	Egypt	900,000
East Indies	3,500,000	Others	220,000

Is it better to represent 100,000 bales by 1 in., or by a smaller length? What length will you use?

6. Bring to school papers or magazines containing graphs of some of the forms studied. Explain what each graph shows. Does the graph show this more clearly than a column of figures would?

Hold a number contest, using operations with whole numbers.



123. Continuous Line Graph.—**Line graphs,** such as the one above, are the most common as well as the most useful graphs. The two heavy lines, at the bottom and at the left, are called the **co-ordinate axes,** or merely the **axes** of the graph. Paper ruled as that in the picture is called **co-ordinate paper.**

EXERCISES

1. State how much is represented in the above graph by one of the squares taken the horizontal way; taken the vertical way.

2. State the population of the United States for the year 1820; 1845; 1885; 1897.

3. State the year in which the population of the United States was 10,000,000; 35,000,000; 25,000,000.

4. Note an irregularity in the curve. What historical reason is there for this?

5. The accompanying readings were taken one winter's day on a Fahrenheit thermometer. To graph the data, first draw the vertical axis to the left on the co-ordinate paper so as to leave 20 squares to the right of it. Next draw the horizontal axis near the bottom of the paper so as to leave at least 40 squares above this line. Let each square along the vertical axis represent 1°. Write along the vertical axis to the left at the proper places 0°, 10°, etc. Let 2 squares along the horizontal axis represent 1 hr. of time. Begin at the intersection of the axes and write below the horizontal axis 8, 9, etc., through 5.

8 A.M.	+ 24°
9 A.M.	+ 28°
10 A.M.	+ 29°
11 A.M.	+ 32°
12 noon	+ 35°
1 P.M.	+ 36°
2 P.M.	+ 35°
3 P.M.	+ 33°
4 P.M.	+ 30°
5 P.M.	+ 27°

Place a mark on the 8-hour line opposite 24°; one on the 9-hour line opposite 28°; and so on. Be very careful to locate each point accurately. Use a sharp pencil to join all the points with a **smooth curve,** as on the opposite page.

6. Suppose the temperature changes were continuous. Read from your graph the temperatures at 8.30; at 11.15; at 3.30; at 2.45.

7. About when was the temperature 25°? 24.5°? freezing? highest?

8. Take readings on your thermometer at home or at the school, at the same time daily for a week or two weeks. Graph your data as you have done the above.

VII

GRAPHS OF EQUATIONS

124. Constants and Variables.—In finding the circumference of a circle, the diameter is multiplied by $\frac{22}{7}$. For each circle the same number, $\frac{22}{7}$, is used. Such numbers are called **constants.** The diameters and circumferences vary for different circles and are called **variables.**

125. Equations with Two Variables.—Equations containing two variables arise often. Such are

$$C = \frac{22}{7}D, \qquad (1)$$

the circumference of a circle in terms of its diameter;

$$v = 32t, \qquad (2)$$

the velocity in feet a falling body gains during t seconds;

$$s = 16t^2, \qquad (3)$$

the space in feet a body falls during t seconds.

126. Independent and Dependent Variables.—As different values are given to t in (2), different values will be obtained for v. The value of v, then, depends upon the value of t. For this reason the literal numbers to which numerical values are assigned, as t, are called **independent variables.** A literal number, as v, whose value depends upon the value *of the other* variable is called the **dependent variable.**

EXERCISES

1. Pick out the constants and the variables in equations 4, 5, and 6 below.

2. Turn to bottom of page 226 and pick out the constants and the variables in each equation.

3. Give five illustrations of independent and dependent variables found in nature or in business.

Fill in the corresponding values in the columns given below:

4. $C = \frac{44}{7}R$

R	C
1	$6\frac{2}{7}$
2	
3	
4	
5	
6	
7	
8	

5. $v = 32t$

t	v
1	32
2	64
3	
4	
5	
6	
7	
8	

6. $s = 16t^2$

t	s
1	16
2	64
3	144
4	
5	
6	
7	
8	

7. Make a continuous line graph from the corresponding numbers found in Ex. 4. Place the *independent variable along the horizontal axis.*

8. From the graph give the radius of the circle whose circumference is 8; 12; 6.5; 20; 15; 12.5.

9. What is the circumference of the circle whose radius is 3.5? 6.25? 7.5? 4.75?

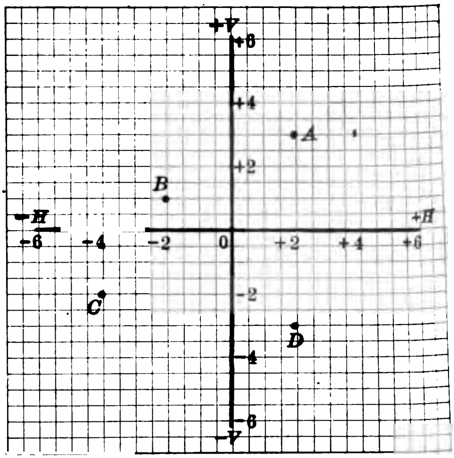
10. Make a continuous line graph from the data in Ex. 5.

11. Find v when t is 6.5; 4.25; 4.75; 5.5.

12. Find t when v is 25; 136; 48; 75.

13. Graph the equation, $P = 3n$.

127. Complete Co-ordinate Diagram.—The point of intersection of the two axes, O, is called the **origin.** Horizontal distances to the right of the vertical axis are positive (+), while horizontal distances to the left are negative (−). Vertical distances above the horizontal axis are positive (+), while distances below are negative (−). A point is located by its distance to the right or to the left,



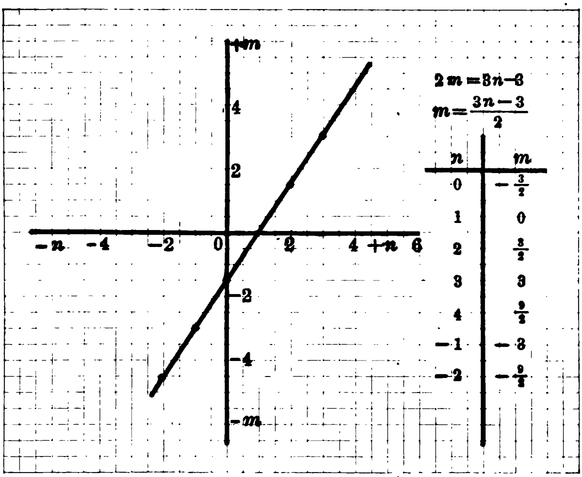
+ or −, of the vertical axis and by its distance above or below, + or −, the horizontal axis. These two distances are called the **horizontal co-ordinate** and the **vertical co-ordinate of the point.** For any point they are written—(3, 4); (5, 1); (4, − 2); (− 3, − 5); (− 4, 3); (− 5, − 2); and so on. State the co-ordinates of the points A, B, C, D, above. Locate in the diagram the points whose co-ordinates have just been given. Suggest and locate other points.

Negative numbers can now be used in graphs as at the top of the next page. Explain fully.

128. Checking Graphs.—All graphs should be checked. To check a graph, select a few points on the graph that were not used in making the graph, as $m = 2$, $n = 2.3$, for the graph upon the opposite page, and substitute these values in the equation. Thus,

$$2 \times 2 = 3 \times 2.3 - 3 = 6.9 - 3, \text{ or } 4 = 3.9,$$

which is as nearly correct as the readings of the co-ordinates of the point could be made.



EXERCISES

1. Select two other points upon the above graph and carry out the check.

2. Find m from the graph when n is 2.5; 3.25; $-$ 2.5; .5.

3. Find n from the graph when m is 2; 3.5 ; $-$ 2; $-$ 1.5.

Graph and check the following equations:

4.	$N = 2 + n$	8.	$3a = 15 - 9b$
5.	$a = b + 5$	9.	$2h = 3g - 4$
6.	$g = h - 5$	10.	$k = 2r - 5$
7.	$2n = 6m + 4$	11.	$5R = 10Q - 15$

12. The readings on the Fahrenheit and the Centigrade thermometers are connected by the equation,

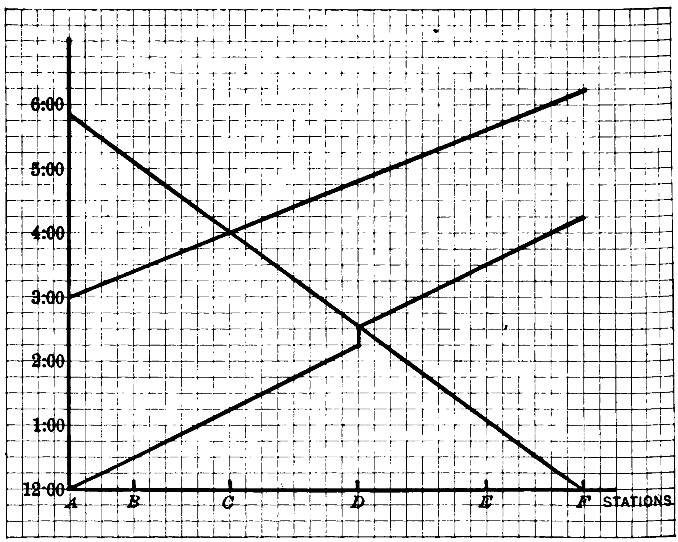
$5F° = 9C° + 160$. Graph the equation.

13. From the graph give C when F is 60°, 150°, 20°, $-$ 40°.

14. From the graph give F when C is 20°, 40°, $-$ 20°.

129. Linear Equations.—It is shown in higher mathematics that the graphs of certain equations will be straight lines. The variables in such equations are of the first degree and both variables do not occur in the same term. How many points determine a straight line? Hence, how many points are needed in graphing these equations? Is it better to select two points near together or far apart for drawing a line? Why? Graph any two equations of Exs. 4–11 on page 133 by the use of only two points and check.

The graph of an equation will be a curve if one or both variables in an equation have exponents greater than 1, or if both variables occur in the same term.



130. Graph of Railroad Time Table.—The above is the graph of a portion of a railroad time table, marked table 7, on the opposite page. Note that one train leaves station *A* at 12 o'clock and another at 3 o'clock, both going toward

station B. A third train leaves station F at 12 o'clock going in the opposite direction. From this graph it is seen at once when any train leaves each of the stations.

The railroad time tables which we all use are made out by means of graphs. These graphs are similar to the one on the opposite page, but are very large. In making out a time table many changes in the running of the trains arise, therefore the black lines in the graph are replaced by threads held down with pins.

TABLE 7

	READ DOWN		READ UP
Sta.	I	III	II
A	12:00	3:00	
B	12:30	3:24	5:07
C	1:15		4:00
D		4:48	2:30
E	3:30	5:36	1:06
F	4:15	6:15	12:00

TABLE 12

	READ DOWN		READ UP
Sta.	I	III	II
A	9:00	11:00	2:00
B	9:40		
C		11:45	12:30
D	10:20	12:00	12:00 11:40
E	11:00	12:30	11:00
F		1:15	
G	1:00	2:00	9:00

EXERCISES

Answer Exs. 1–4 from the graph on the opposite page.

1. When does train II reach A? When does train III reach C?

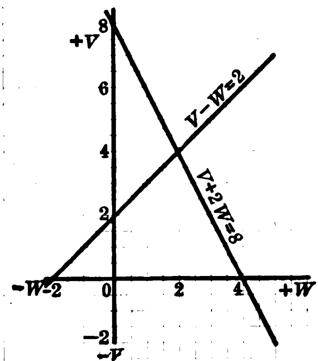
2. When does train I arrive at D? When does it leave D? Why does it wait at D?

3. When and where do trains II and III meet?

4. How long does it take each train to go between stations A and D? Which is the fastest?

5. Make a graph of table 12. Fill in the vacant places from the graph.

6. Try to give the graphs of some trains from a railroad time table. Ask and answer questions like those here given.



$V+2W=8$
$V=8-2W$

W	V
0	8
1	6
2	4
3	2
4	0
5	−2
−1	10

$V-W=2$
$V=2+W$

W	V
0	2
1	3
2	4
3	5
4	6
5	7
−1	1
−2	0

131. Equations Graphed Together.—The above gives the graph of each of the equations,

$$V + 2W = 8, \qquad (1)$$
$$V - W = 2. \qquad (2)$$

EXERCISES

1. From the graph tell what values of V for each equation will correspond to the following values of W: 1, 3, 5, 0, −.5, 3.5, − 1.

2. Find from the graph the value of V and the corresponding value of W that will be the same in both equations.

3. Substitute these values of V and W in both equations to check the work. What do you find?

132. Simultaneous Equations.—Equations taken together, as those above, are called **simultaneous equations.** To solve a pair of simultaneous equations means to find *a pair of values* for the variables which satisfies both equa-

tions. By the graph on page 136 it was found that $V = 4$ and $W = 2$ satisfied both of the equations,

$$V + 2W = 8,$$
$$V - W = 2.$$

The solution of these two equations is therefore $V = 4$, $W = 2$.

The graphs of two equations may give the same line; then the co-ordinates of all points satisfying one equation will also satisfy the other equation. To which of the following exercises does this apply?

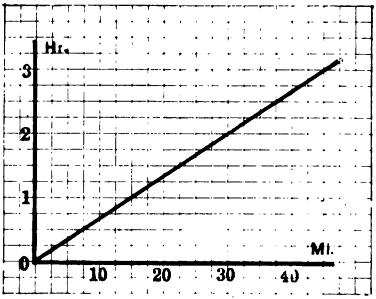
The graphs of two equations may be two parallel lines; the equations then have no solution. Why? To which of the following exercises does this apply?

EXERCISES

Graph the following sets of equations to the same axes, find their point of intersection, and check:

1. $x + 2y = 8$
   $x - y = 2$
2. $m - n = 2$
   $m + n = 6$
3. $3x - z = 14$
   $2x + z = 11$
4. $g + 2h = 1$
   $3g - h = 10$
5. $3a = b + 1$
   $a = 2b - 3$
6. $5p = 3q + 7$
   $2p = q + 3$
7. $3r - 2w = 7$
   $2r - w = 4$
8. $3c + 7d = 1$
   $2d - c = 4$
9. $a + b = 6$
   $2a + 2b = 12$
10. $m + 2n = 4$
    $m + 2n = 6$

11. Express the cost, C, of P pounds of apples at 5 ¢ per pound. Graph the equation. If delivered, 10 ¢ is added to the total cost. Express this as an equation and graph. How do the graphs run? How do the two values of P vary for each value of C? How do the two values of C vary for each value of P?



John made a trip on a motor-cycle at the rate of 15 mi. per hour. The graph shows his motion.

12. Henry started in an auto along the same road at the rate of 25 mi. an hour, two hours after John. Make a larger graph of John's motion. Graph Henry's motion to the same pair of axes. At what point on the time axis will the graph of Henry's motion begin?

From the graph answer the following:

13. How far did Henry travel in 5 hr.? in 3 hr.? in $\frac{1}{2}$ hr.?

14. In what time did John travel 12 mi.? 23 mi.? 18 mi.?

15. In what time did Henry travel 15 mi.? 30 mi.?

16. How long does it take Henry to overtake John?

133. Solution of Simultaneous Equations by Substitution.

$$3v + 2w = 12 \qquad (1)$$

$$5v - 3w = 1. \qquad (2)$$

If equation (1) is solved for w, then,

$$w = \frac{12 - 3v}{2}. \qquad (3) \quad \text{How?}$$

Since (1) and (2) are to have the same value of v for the same value of w, the value of w found in (3) may be substituted in (2) when

$$5v - \frac{3(12 - 3v)}{2} = 1. \qquad (4)$$

Hence,

$$10v - 36 + 9v = 2 \qquad (5) \quad \text{How?}$$

$$19v = 38 \qquad (6) \quad \text{How?}$$

$$v = 2. \qquad (7) \quad \text{How?}$$

Substituting this value of v in (3) gives

$$w = \frac{12 - 6}{2} \quad (8)$$

$$= 3. \quad (9) \text{ How?}$$

Hence, the solution of (1) and (2) is $v = 2$, $w = 3$. Check in (1) and (2).

The value of v could just as well have been found in terms of w and that substituted in the other equation, so as to have obtained a single equation in w. Do this and check.

EXERCISES

Solve by substitution and check:

1. $r + s = 7$
   $2r + 3s = 16$
2. $x + y = 4$
   $17x - 5y = 2$
3. $a + 2b = 4$
   $2a + b = 5$
4. $2m - 7n = 7$
   $m + 8n = 15$
5. $5h - 10k = 25$
   $h - 4k = 1$
6. $3r + 2s = 5$
   $5r - 3s = 2$
7. $7g + 6h = 32$
   $5g - 3h = 1$
8. $3c - 2d = 15$
   $5c + 6d = 81$
9. $2x + 3y = 15$
   $4x + 9y = 50$
10. $4A + 3B = 16$
    $8A - 9B = -3$

Solve the following exercises graphically:

11. The sum of two numbers, M and N, is 13. Twice M plus N equals 21. Set up the two equations and solve.

12. The sum of two numbers is 18. The larger minus the smaller equals 2. What are the numbers?

13. The base and the altitude of a rectangle are represented respectively by b and a. Set up one equation stating that the perimeter equals 42 in. and another that the base minus the altitude equals 1 in. Find base and altitude.

14. The perimeter of a rug is 44 ft. The length is 2 ft. more than the width. What are the dimensions of the rug?

15. A geometry and a botany cost \$ 3. The botany costs 50 ¢ more than the geometry. Find the cost of each.

Hold a number contest, using operations with decimals.

134. Solution of Simultaneous Equations by Addition or Subtraction.—In the equations,

$$v + 2w = 8, \qquad (1)$$
$$v - w = 2, \qquad (2)$$

if the second be multiplied by 2, then,

$$v + 2w = 8, \qquad (1)$$
$$2v - 2w = 4. \qquad (3)$$

By adding (1) and (3) w drops out and

$$3v = 12, \qquad (4) \quad \text{How?}$$
$$v = 4. \qquad (5) \quad \text{How?}$$

This value of v may be substituted in either (1) or (2) and the corresponding value of w found. Thus,

$$4 + 2w = 8. \qquad (6) \quad \text{How?}$$

Hence,
$$2w = 4, \qquad (7) \quad \text{How?}$$
$$w = 2. \qquad (8) \quad \text{How?}$$

These values of v and w are now substituted in (2), the equation which was not used in finding w, to check the work. Make this substitution.

The equations could also have been solved by merely subtracting (2) from (1), when

$$3w = 6, \qquad (9) \quad \text{How?}$$
$$w = 2. \qquad (10) \quad \text{How?}$$

Combining equations so as to get an equation without some one literal number is called **elimination** of that number. In the above, w was eliminated first and then v.

EXERCISES

Solve by the processes on page 140 and check:

1. $x + 2y = 8$
   $3x - 5y = -9$

2. $2g + 3h = 5$
   $7g - 5h = 33$

3. $a - b = 1$
   $3a - 5b = -13$

4. $4c + 3d = 5$
   $3c + 7d = 0$

5. Solve Exs. 4 and 5 on page 139 by this method.

6. Solve Exs. 6 and 7 on page 139 by this method.

7. Solve Exs. 8 and 9 on page 139 by this method.

8. Solve Exs. 10 and 11 on page 139 by this method.

9. Find two numbers such that 3 times the first plus the second equals 46, and 5 times the first minus 4 times the second equals 20.

10. A school has 324 pupils. If the number of girls equals twice the number of boys, how many boys and girls are there in the school?

11. A house and lot cost $ 8000. If the house cost $ 2500 more than the lot, what did each cost?

12. In a well-proportioned living-room the width is 75 % of the length. What are to be the dimensions of a living-room so that it may be 6′ longer than it is wide?

13. James and Harry together have 22 Baby Bonds. If James had 1 less and Harry had 3 more, each would have the same number. How many Baby Bonds has each?

14. Mary and Jane have saved their money to buy Thrift Stamps. At the end of one month they together had 15 stamps. Jane had 3 stamps more than Mary. How many Thrift Stamps had each?

135. Other Forms of Simultaneous Equations.—Equations in two variables not in the form just studied can easily be changed to other true equations of this form. Thus,

$$\frac{3}{a-x} = \frac{5}{a+x}, \qquad (1)$$

$$\frac{a+2x}{3} + \frac{a+3x}{7} = 3. \qquad (2)$$

From (1) $\quad 3a + 3x = 5a - 5x. \quad$ (3) How?

From (2) $\quad 7a + 14x + 3a + 9x = 63. \quad$ (4) How?

From (3) $\quad -2a + 8x = 0. \quad$ (5) How?

From (4) $\quad 10a + 23x = 63. \quad$ (6) How?

Complete the solution and check in (1) and (2).

EXERCISES

Solve and check the following:

1. $a - 4b = \dfrac{4a-28}{5}$
   $\dfrac{a+5b}{2} = \dfrac{2a+9}{3}$

2. $\dfrac{r+s}{7} - 1 = \dfrac{s-r}{2}$
   $\dfrac{r+s}{2} - \dfrac{r-s}{3} = \dfrac{23}{3}$

3. $\dfrac{m+n-5}{3} = 1$
   $\dfrac{2m+4n+2}{5} = 5$

4. $\dfrac{a+2b}{3} + 2b = \dfrac{17}{3}$
   $\dfrac{2a-1}{5} + 2b = 3$

5. $\dfrac{3}{m} + \dfrac{2}{n} = \dfrac{28}{mn}$
   $\dfrac{5}{m} - \dfrac{2}{n} = \dfrac{44}{3mn}$

6. $\dfrac{5a}{2} + 4b = 2\frac{1}{3}$
   $a + 5b = -4\frac{1}{6}$

7. Write a formula stating that a number of two digits may be expressed by 10 times the first digit plus the second digit.

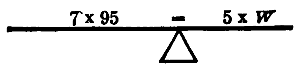
8. Find the two digits of a number whose sum is 12 and the unit's digit is twice the ten's digit.

9. If 9 be added to a certain number, it will give another number with the same digits in reversed order. The sum of the two digits is 11. What is the number?

10. John and Henry together own \$ 350 in Baby Bonds. Twice what Henry owns is \$ 100 more than what John owns. What is the value of the bonds owned by each of the boys?

11. Alice and her sister Mabel together have bought 24 Thrift Stamps. If Alice had bought 2 more, they would have had the same number. How many Thrift Stamps did each girl buy?

12. Two boys play on a teeter-board 12 ft. long. The support is called the fulcrum. One boy weighing 95 lb. sits 7 ft. from the fulcrum, while the other boy sits 5 ft. from the fulcrum. What is the second boy's weight? Experiments have shown that to obtain a balance, the weight of one times his distance from the fulcrum equals the weight of the other times his distance from the fulcrum.



13. Sara weighs 64 lb. and sitting on a teeter-board 4 ft. 6 in. from the fulcrum balances Hattie, who weighs 85 lb. How far from the fulcrum does Hattie sit?

14. If Sara moves so as to be 5 ft. 3 in. from the fulcrum, where must Hattie sit so as to balance?

15. The sum of two angles is 120°. The larger equals the smaller increased by 60°. How many degrees are there in each angle?

16. Express by an equation the distance, d, a freight-train has gone in t hours at 15 mi. per hour. How long has a passenger-train been going that left 3 hr. later? Express the distance it has gone at 25 mi. per hour. Where does the passenger-train overtake the freight?

136. Simultaneous Equation with Three Variables.—Equations with three variables cannot be graphed in a plane. Some simultaneous equations with three or more variables of the first degree can be solved by substitution, addition, or subtraction. Thus,

$$2a + 3b - c = 18, \qquad (1)$$
$$3a + 5b + 2c = 25, \qquad (2)$$
$$4a - 2b - 5c = 15. \qquad (3)$$

Multiplying (1) by 2 we get

$$4a + 6b - 2c = 36. \qquad (4)$$

Adding (4) and (2) gives

$$7a + 11b = 61. \qquad (5)$$

Multiplying (1) by 5 gives

$$10a + 15b - 5c = 90. \qquad (6)$$

Subtracting (3) from (6) gives

$$6a + 17b = 75. \qquad (7)$$

Solving (5) and (7) by any of the methods given previously gives $a = 4$ and $b = 3$. Test in (5) and (7).

c is found by substituting the values found for a and b in one of the given equations. Substituting in equation (1) gives

$$8 + 9 - c = 18, \qquad (8)$$

hence,

$$c = -1. \qquad (9)$$

The work is verified by substituting the values found for a, b, and c in the two equations not used in finding c. Substituting in (2) gives

$$12 + 15 - 2 = 25. \qquad (10)$$

Was the work correct? Verify by substituting in equation (3).

EXERCISES

1. Solve the above equations by first eliminating b from two pairs of the equations, thus obtaining two equations in a and c.

2. Solve the same set of equations by eliminating a from two pairs of equations, thus obtaining two equations in c and b.

Solve and check the following:

3. $$\begin{aligned} 2r + 3s + 4t &= 9 \\ r + s - t &= 1 \\ 3r + 5s - 7t &= 1 \end{aligned}$$

4. $$\begin{aligned} m + n + 2r &= 3 \\ 2m + 3n - r &= 19 \\ 5m - 5n + 3r &= -1 \end{aligned}$$

5. $$\begin{aligned} d + e + f &= 7 \\ 2d + 4e - 2f &= 5 \\ 5d + 3e - 3f &= 9 \end{aligned}$$

6. $$\begin{aligned} x + 3y &= 18 \\ 2x + y + z &= 13 \\ 2y + 3z &= 16 \end{aligned}$$

7. $$\begin{aligned} g + h &= 22 \\ h - k &= 17 \\ g + 2k &= 0 \end{aligned}$$

8. $$\begin{aligned} \frac{a+b}{3} + \frac{c+2}{6} &= 4 \\ \frac{a-c}{2} + \frac{b+c}{3} &= \frac{7}{2} \\ a + b + c &= 12 \end{aligned}$$

9. John, William, and Harry together carry 165 papers. John and William together carry 21 more than Harry. Three times the number that William carries equals twice the number that Harry carries. How many papers does each boy carry?

10. James raised strawberries for market. For the earliest berries he received 13 ¢ per quart, for the next earliest 10 ¢ per quart, and for the later ones only 7 ¢ per quart. He sold 80 qt. altogether. His total receipts were $7.70. If he sold as many quarts at 10 ¢ as at the other two prices together, how many quarts of each kind did he sell?

11. Madge, Cora, and Ethel together weigh 164 lb. Cora's weight added to twice Madge's weight lacks 8 lb. of being 4 times Ethel's weight. Twice Ethel's weight added to Madge's weight equals twice Cora's weight plus 49 lb. Find the weight of each one of the girls.

Hold a number contest on solving simultaneous equations.

VIII

THEORY OF EXPONENTS AND RADICALS

137. Review of Laws of Exponents.—Review thoroughly the laws of exponents previously studied in Arts. 40 and 59 and apply them to the following exercises.

EXERCISES

Carry out the following operations:

1. $3a^3 \times 5a^2$
2. $b^4 \times 6b^3$
3. $2a^4 \times 3a^5$
4. $2a^2b \times 3ab^2$
5. $(-3mn^3) \times 5mn^2$
6. $m^5 \div m^3$
7. $a^9 + a^5$
8. $15m^{10} \div 3m^3$
9. $-27m^3 \div (-9m^2)$
10. $8a^4b \div 2a^3b$
11. $a^4b^4 \div a^2b^3$
12. $21m^{10}n^5 \div 7m^2n^4$
13. $75a^3d^9 \div (-25a^2d^5)$
14. $t^3u^8 \div tu^5$
15. $77r^3s^2t^3 \div 7rst^2$
16. $a^3b(3ab^3 - 9a^2b^4 + b^5)$
17. $(12a^4b^4c^4 - 8a^5b^5c) \div 4b^4$
18. $(44x^5 - 77x^4 + 99x^8) \div (-11x^2)$
19. $(3r^2s + 7rs^2 - 9) \times (-8r^2s^3)$
20. $(h^{10} - 2h^8g^8 + g^{10}) \times g^2h^3$
21. $(m^4 - 2m^8n^4 + m^4n^4) \div m^4$
22. $(3t^2u + 3tu^2 + u^3) \div u$
23. $(64x^3y^3 + 56x^8y^2) \div (-8x^3y^2)$
24. $(-49g^3k^2 - 14g^5k^7) \div (-7g^3k^2)$
25. $(144a^4b^4 - 24a^5b^7c) \div 12a^4b^3$
26. $(532x^4z^5 - 96x^7z^{12}) \div (-4x^3z^5)$
27. $-7r^3s^2(-8ar^2 + 9bs^3 - 1)$
28. $(8a^4b^3 - 8ab^4 + 16ab^5) \div 8ab^3$
29. $(p^3q^3 - 20p^4q^4 - p^8q^7) \div p^2q^3$
30. $-17a(-3a^3b - 4ab^3 + c)$

138. Fractional Exponents.—Since $m \times m \times m = m^3$, m is one of the three equal factors of m^3. In the same way, $m^{\frac{1}{3}} \times m^{\frac{1}{3}} \times m^{\frac{1}{3}} = m$, or $m^{\frac{1}{3}}$ is one of the three equal factors of m. Similarly $8^{\frac{1}{3}} \times 8^{\frac{1}{3}} \times 8^{\frac{1}{3}} = 8$. Hence, $8^{\frac{1}{3}}$ is one of the three equal factors of 8. Then $8^{\frac{1}{3}} = 2$, for 2 is one of the three equal factors of 8. One of these equal factors is called a **root.** Finding a root is just the reverse of finding a power. Corresponding to square, cube, fifth power, and so on there are **square root, cube root, fifth root,** and so on. A fractional exponent whose numerator is 1 indicates that the root stated by the denominator is to be taken. Thus, $125^{\frac{1}{3}} = 5$; $32^{\frac{1}{5}} = 2$; $81^{\frac{1}{2}} = 9$; $81^{\frac{1}{4}} = 3$; $(36a^4k^2)^{\frac{1}{2}} = 6a^2k$; $(125a^6d^3)^{\frac{1}{3}} = 5a^2d$; etc.

EXERCISES

Carry out the operations indicated:

1. $4^{\frac{1}{2}}$	**7.** $16^{\frac{1}{2}}$	**13.** $(a^2b^4)^{\frac{1}{2}}$	**19.** $(49q^2)^{\frac{1}{2}}$
2. $25^{\frac{1}{2}}$	**8.** $1^{\frac{1}{2}}$	**14.** $(x^4y^6)^{\frac{1}{2}}$	**20.** $(16x^4)^{\frac{1}{4}}$
3. $8^{\frac{1}{3}}$	**9.** $1^{\frac{1}{5}}$	**15.** $(8s^6t^3)^{\frac{1}{3}}$	**21.** $(27k^3q^6)^{\frac{1}{3}}$
4. $27^{\frac{1}{3}}$	**10.** $125^{\frac{1}{3}}$	**16.** $(27t^6r^9)^{\frac{1}{3}}$	**22.** $(64v^6w^6)^{\frac{1}{2}}$
5. $216^{\frac{1}{3}}$	**11.** $343^{\frac{1}{3}}$	**17.** $(a^4c^8)^{\frac{1}{4}}$	**23.** $(64v^6w^6)^{\frac{1}{3}}$
6. $16^{\frac{1}{4}}$	**12.** $100^{\frac{1}{2}}$	**18.** $(z^{10}y^{15})^{\frac{1}{5}}$	**24.** $(100p^4q^6)^{\frac{1}{2}}$

25. If S is the surface of a sphere and R its radius, then

$$S = 4\pi R^2, \qquad (1)$$

when $R = [S \div 4\pi]^{\frac{1}{2}}$. How? (2)

Using π as $\frac{22}{7}$, find R when S is $\frac{352}{7}$ sq. in.; 154 cm.^{2}; $\frac{1782}{7}$ sq. ft.

Hold a number contest on multiplying and dividing literal numbers.

139. Radical Signs.—Roots are also indicated by the **radical sign** $\sqrt{\ }$, as we have seen in square root. The required root is indicated by a small numeral in the $\surd$ part of the sign, except that no numeral is used when the square root is indicated. Thus, $\sqrt{16} = 4$; $\sqrt[3]{125} = 5$; $\sqrt{36a^4b^2} = 6a^2b$; $\sqrt[4]{16m^8n^4} = 2m^2n$; etc.

EXERCISES

Carry out the indicated operations:

1. $\sqrt{49}$	6. $\sqrt{9a^2}$	11. $\sqrt{25x^6z^2}$	16. $\sqrt{36m^2n^2}$
2. $\sqrt{100}$	7. $\sqrt[3]{27a^3}$	12. $\sqrt{100x^4y^6}$	17. $\sqrt{a^2b^2c^4}$
3. $\sqrt{a^2}$	8. $\sqrt[3]{125r^3}$	13. $\sqrt[3]{64x^6y^{12}}$	18. $\sqrt[3]{a^9b^6c^3}$
4. $\sqrt[3]{8}$	9. $\sqrt[4]{16a^4b^4}$	14. $\sqrt{64x^6y^{12}}$	19. $\sqrt{81r^4s^6}$
5. $\sqrt[3]{27}$	10. $\sqrt[5]{32n^{10}m^{15}}$	15. $\sqrt{49y^8z^4}$	20. $\sqrt{144q^8r^6}$

21. The exponential form of expressing roots is much clearer and is the one you should get into the habit of using. Change the above to the form using exponents.

22. If s is the length of the side of a cube, express its volume in terms of s. Express its side, s, in terms of its volume, v.

23. Find the side of the cube whose volume is 216 cu. in.; 125 m.^{3}; 64 cu. ft.

24. The number of gallons a cylindrical pipe or tank will hold is given by the formula,

$$G = D^2 \times L \times .0408,$$

where D is in inches and L in feet. Solve for D.

25. How many gallons will a tank hold that has a height of 8 ft. and a diameter of 14 in.?

26. Find the diameter of a tank 6 ft. high which holds 80 gal.

140. Extension of Fractional Exponents.—$m^{\frac{1}{4}} \times m^{\frac{1}{4}} \times m^{\frac{1}{4}} = m^{\frac{3}{4}}$ shows that the root indicated by the denominator is to be taken and that result raised to the power indicated by the numerator. Thus,

$$(25a^4b^2)^{\frac{3}{2}} = [(25a^4b^2)^{\frac{1}{2}}]^3 = [5a^2b]^3 = 125a^6b^3;$$
$$(9)^{\frac{3}{2}} = [3]^3 = 27;\ (27)^{\frac{2}{3}} = (3)^2 = 9.$$

EXERCISES

Carry out the indicated operations:

1. $25^{\frac{3}{2}}$	6. $27^{\frac{2}{3}}$	11. $16^{\frac{3}{4}}$	16. $4^{\frac{5}{2}}$
2. $8^{\frac{2}{3}}$	7. $16^{\frac{3}{2}}$	12. $49^{\frac{3}{2}}$	17. $8^{\frac{5}{3}}$
3. $4^{\frac{3}{2}}$	8. $36^{\frac{3}{2}}$	13. $27^{\frac{5}{3}}$	18. $32^{\frac{2}{5}}$
4. $1^{\frac{2}{3}}$	9. $100^{\frac{3}{2}}$	14. $144^{\frac{3}{2}}$	19. $64^{\frac{4}{3}}$
5. $27^{\frac{4}{3}}$	10. $125^{\frac{2}{3}}$	15. $1000^{\frac{2}{3}}$	20. $81^{\frac{3}{2}}$

21. If l is the length of the edges of a cube, what is its volume? From this we have

$$l = \sqrt[3]{V} = (V)^{\frac{1}{3}}.$$

How many faces has a cube? What is the area of one face? of all the faces? Hence, the total surface will be

$$S = 6(V)^{\frac{2}{3}}.$$

Find the surface of the cube whose volume is 125 cu. in.

22. A part of an electrical formula is

$$[r^2 + d^2]^{\frac{3}{2}}.$$

Find its value when $r = 4$, $d = 3$; $r = 5$, $d = 12$.

23. The formula for the volume of a cone is

$$V = \tfrac{1}{3}\pi r^2 h.$$

Express r in terms of the other quantities.

Hold a number contest on raising several numbers to powers and extracting roots.

141. Square Root of a Polynomial.—Squaring $a + b + c$ gives $a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$, which can be written in the form

$$a^2 + (2a + b)b + (2a + 2b + c)c.$$

In finding the square root of a polynomial, it is first written so that the exponents of some one letter are increasing or decreasing in regular order, as shown in the illustration.

	$2m + 3n - 5h$
	$4m^2 + 12mn - 20mh - 30nh + 9n^2 + 25h^2$
$a \times a$	$4m^2$
$2ab \div 2a = b$ (here $3n$)	$12mn + 9n^2$
$(2a + b) \times b = 2ab + b^2$ (here $12mn + 9n^2$)	$12mn + 9n^2$
$2ac \div 2a = c$ (here $-5h$)	$-20mh - 30nh + 25h^2$
$(2a + 2b + c)c = 2ac + 2bc + c^2$ (here $-20mh - 30nh + 25n^2$)	$-20mh - 30nh + 25h^2$

The square root of the first term is found and placed above the polynomial. This is then squared and subtracted from the polynomial and the next term, $12mn$, brought down. The part brought down is of the form $2ab$. Dividing by $2 \times 2m$, it gave the second term of the root, $3n$. The first term in the root doubled, plus the second term, takes the form $2a + b$. This sum multiplied by the second term, $3n$, gives $12mn + 9n^2$ to be subtracted from the polynomial. The next term containing the first term in the root, $-20mh$, is then divided by twice the first term in the root, giving $-5h$ for the third term in the root. Multiplying the part already found by 2 and adding in the third term gives $4m + 6n - 5h$. Now multiplying by the third term, $-5h$, and subtracting from the polynomial leaves no remainder. This shows that $2m + 3n - 5h$ is the square root of the polynomial $4m^2 + 12mn - 20mh - 30nh + 9n^2 + 25h^2$.

EXERCISES

1. How many square roots has a number? What are they for 4? for 25? for 121? for 36? for 216? for $9h^4$? for $25k^2m^6$? for $(w + q)^2$? for $[l + 2s]^2$? for $m^2 + 2mn + n^2$? for $f^2 + 6fg^2 + 9g^4$?

2. The two square roots of a number may be indicated by a parenthesis preceded by $\pm$, meaning that one root has the sign $+$ and the other the sign $-$. Thus, the square root of $9x^2 - 6x + 1$ is $\pm(3x - 1)$. What are the two square roots of $16x^2 - 8x + 1$?

Find the square roots of each of the following and check your work by squaring each. Solve through Ex. 12 without pencil and paper.

3. $a^2 - 10a + 25$
4. $x^2y^2 - 12xy + 36$
5. $m^2 - 8m + 16$
6. $9k^2 - 6k + 1$
7. $25z^2 - 20z + 4$
8. $36a^2 - 84a + 49$
9. $9x^2y^2 - 30xy + 25$
10. $g^4 - 6g^2 + 9$
11. $a^6 - 12a^3b^3 + 36b^6$
12. $k^8y^4 - 20k^4y^2 + 100$
13. $g^2 + 4gh + 6gk + 4h^2 + 12hk + 9k^2$
14. $4a^4 + 8a^3 + 8a^2 + 4a + 1$
15. $9x^4 - 12x^3 + 10x^2 - 4x + 1$
16. $4z^4 - 16z^3 + 36z^2 - 40z + 25$
17. $-20t + 14t^2 + 25 + t^4 - 4t^3$
18. $a^2 + b^2 + w^2 + 2ab - 2aw - 2bw$
19. $g^2 + 2gh - 8gk + h^2 - 8hk + 16k^2$
20. $9k^2 + 25h^2 + 49 + 30hk + 42k + 70h$
21. $4v^6 - 12v^5 + 9v^4 + 4v^3 - 6v^2 + 1$
22. $a^2b^2 + 2abcd + c^2d^2 + 6ab + 6cd + 9$
23. $4a^4b^4 - 12a^3b^3 + 37a^2b^2 - 42ab + 49$
24. $9r^2 + 4s^2 + t^2 - 12rs - 6rt + 4st$
25. $a^2 + b^2 + c^2 + d^2 + 2ab + 2ac + 2ad + 2bc + 2bd + 2cd$
26. $58r^3 - 52r^2 - 28r + 9r^6 - 24r^5 + 4r^4 + 49$
27. $25x^4 + 29x^2 + 4 - 30x^3 - 12x$

Hold a number contest on extracting square roots of polynomials.

142. Approximate Roots of Arabic Numbers.—The square of every number will have either twice as many digits or one less than twice as many digits as there are in the number squared. Is this true for 56^2 and for 235^2? Hence, it is necessary first to decide how many digits there are to be in the root to be found. This is done by beginning at the decimal point and dividing the number into periods of 2 digits in each direction. The number 55327.56 takes the form 5′53′27.′56′, showing that its square root will have 3 digits in the whole number. Follow very carefully the form below in order to avoid confusion and to get the correct result quickly.

143. Finding the Square Root of 55327.56.—After dividing the number into periods, as above, the work takes the following form, which is similar to the one on page 150.

a is really 20 but zeros are omitted.		2 3 5. 2 1	
		5′53′27.′56′	
		4	a^2
$2a$ or $2 \times 20 = 40$	40	153	$2ab$ + other part.
b is 3	3		
$2a + b$ is $2 \times 20 + 3$	43	129	$(2a + b) \times b$
$2a$ is 2×230	460	2427	$2ab$ + other part.
b is now 5	5		
$2a + b$ is $2 \times 230 + 5$	465	2325	$(2a + b) \times b$
$2a$ is 2×2350	4700	10256	$2ab$ + other part.
b is now 2	2		
$2a + b$ is $2 \times 2350 + 2$	4702	9404	$(2a + b) \times b$
$2a$ is 2×23520	47040	85200	$2ab$ + other part.
b is now 1	1		
$2a + b$ is $2 \times 23520 + 1$	47041	47041	$(2a + b) \times b$
		38159	

The largest digit whose square is not larger than the first period is placed as the first digit in the root. This is then squared, its square *subtracted* from the first period, and the next period brought down.

A zero is now annexed to the root already found and that number doubled. This is then divided into the part of the number worked with to obtain the next digit in the root. The next digit in the root is then added to this number and its sum multiplied by the second digit of the root. This product is subtracted from the part of the number worked with and the next period brought down. The process of finding the second digit now repeats.

By annexing periods of two zeros after any decimal, or to the right of the decimal point in whole numbers, roots may be extracted to any number of decimal places.

144. Checks.—Check the work by squaring the root found and add in the remainder, if any. This will give the number whose root is sought if the work is correct.

EXERCISES

Find the square root of the first 16 exercises to 2 decimal places and check the work:

1. 523.24	**5.** 0.4035	**9.** 1.8424	**13.** 2
2. 52.38	**6.** 0.3636	**10.** 8.3467	**14.** 3
3. 13.45	**7.** 5.3425	**11.** 34.0536	**15.** 5
4. 32.0564	**8.** 1.234	**12.** 823.4	**16.** 7

Use the results from Exs. 13–16 to solve the following equations. Check your results:

17. $a^2 = 2$	**21.** $m^2 - 1 = 6$
18. $3m^2 = 15$	**22.** $3g^2 + 2 = 17$
19. $7k^2 = 49$	**23.** $4w^2 + 1 = 16 - w^2$
20. $8t^2 = 24$	**24.** $(a + 1)^2 = 2a + 8$

25. From formula (2) in Ex. 25 on page 147 find the radius of the sphere whose area is 792 sq. in.

26. From the formula in Ex. 22 on page 149 find the value of the expression when $r = 3$ and $d = 5$.

27. From the formula in Ex. 23 on page 149 find the value of r for $V = 616$ cu. in. and $h = 12$.

145. Approximate Roots in Equations.—The solution of quadratic equations often involves approximate roots. Thus,

$$a^2 - 2a = 2 \qquad (1)$$

$$a^2 - 2a + 1 = 3 \qquad (2) \quad \text{How?}$$

$$a - 1 = \pm 1.732 \qquad (3) \quad \text{How?}$$

$$a = 2.732 \text{ or } -.732. \qquad (4) \quad \text{How?}$$

Checking: substituting 2.732 in (1) gives

$$7.46 - 5.46 = 2 \qquad (5) \quad \text{How?}$$

or

$$2.00 = 2. \qquad (6) \quad \text{What does this show?}$$

Carry out the check by substituting -0.732 in (1).

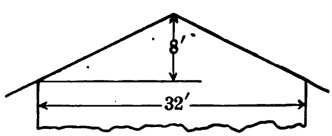
EXERCISES

Find the roots of the following to two decimal places and check:

1. $t^2 - 4t = 1$
2. $4r^2 + 4r = 6$
3. $25a^2 - 10a = 2$
4. $g^2 - 2g = 6$
5. $16k^2 - 16k = 1$
6. $36r^2 - 12r = 24$
7. $4p^2 - 16p = -9$
8. $m^2 - 18m = 119$
9. $49d^2 - 28d = -1$
10. $h^2 - 8h = 284$

11. The square on the hypotenuse of a right triangle equals the sum of the squares on the other two sides. Express this by a mathematical equation.

12. A carpenter wishes to make the span of the gable ends of a porch 32 ft. and the rise 8 ft. What length of rafters must he use?



13. The area of a circle is 31.416 sq. in. Find the radius to two decimal places, using π as 3.1416.

Hold a number contest on solving quadratic equations.

146. Roots of Fractions.—To extract a root of a fraction means finding that root of each term of the fraction. Thus, $(\frac{4}{9})^{\frac{1}{2}} = \frac{2}{3}$. It is, however, usually impossible to find the roots of each term of the fraction. Sometimes the fraction can be reduced to a simple decimal of which the required root can be found. Thus, $(\frac{2}{5})^{\frac{1}{2}} = (.4)^{\frac{1}{2}} = ?$

When difficult to reduce the fraction to a decimal, as in $(\frac{2}{3})^{\frac{1}{2}}$, multiply both terms of the fraction by such a number that the required root of the denominator can be found. Thus, $(\frac{2}{3})^{\frac{1}{2}} = (\frac{2}{3} \times \frac{3}{3})^{\frac{1}{2}} = (\frac{6}{9})^{\frac{1}{2}} = \frac{(6)^{\frac{1}{2}}}{3}$. The root of 6 is now found and that divided by 3. Similarly,

$$(\tfrac{3}{7})^{\frac{1}{2}} = (\tfrac{3}{7} \times \tfrac{7}{7})^{\frac{1}{2}} = \frac{(21)^{\frac{1}{2}}}{7}.$$

Roots of fractions which cannot be found at once are in their simplest form when they have been reduced to equivalent fractions in which the root appears only in the numerator.

EXERCISES

Change the following to their simplest form:

1. $(\frac{25}{36})^{\frac{1}{2}}$
2. $(\frac{8}{125})^{\frac{1}{3}}$
3. $(\frac{1}{8})^{\frac{1}{3}}$
4. $(\frac{1}{4})^{\frac{1}{3}}$
5. $(\frac{1}{27})^{\frac{1}{3}}$
6. $(\frac{1}{9})^{\frac{1}{3}}$
7. $(\frac{36}{49})^{\frac{1}{2}}$
8. $(\frac{64}{25})^{\frac{1}{2}}$
9. $(\frac{1}{5})^{\frac{1}{2}}$
10. $(\frac{8}{7})^{\frac{1}{2}}$
11. $(\frac{1}{4})^{\frac{1}{3}}$
12. $(\frac{2}{9})^{\frac{1}{3}}$
13. $(\frac{3}{8})^{\frac{1}{2}}$
14. $(\frac{3}{5})^{\frac{1}{2}}$
15. $(\frac{5}{12})^{\frac{1}{2}}$
16. $(\frac{1}{2})^{\frac{1}{2}}$
17. $(\frac{7}{20})^{\frac{1}{2}}$
18. $(\frac{5}{16})^{\frac{1}{3}}$
19. $(\frac{49}{121})^{\frac{1}{2}}$
20. $(\frac{1}{16})^{\frac{1}{3}}$
21. $(\frac{8}{5})^{\frac{1}{3}}$
22. $(\frac{5}{64})^{\frac{1}{2}}$
23. $(\frac{8}{11})^{\frac{1}{2}}$
24. $(\frac{5}{8})^{\frac{1}{3}}$

Solve and verify the equations:

25. $3f^2 = \frac{4}{3}$
26. $7m^2 = \frac{28}{25}$
27. $16d^2 = 49$
28. $5e^2 = \frac{9}{5}$

147. Square Roots of Fractions in Equations.—Quadratic equations often involve square roots of fractions in their solution. Thus,

$$9a^2 + 15a = 6. \qquad (1)$$

To complete the square, proceed as in Art. 81; the number to square and to add to both members of the equation is $\frac{15a}{6a} = \frac{5}{2}$. How?

Hence, $\quad 9a^2 + 15a + \frac{25}{4} = 6 + \frac{25}{4}$, or $\frac{49}{4}$. $\quad$ (2)

Hence, $\quad 3a + \frac{5}{2} = \pm \frac{7}{2}$ $\quad$ (3) How?

and $\quad 3a = 1$ or -6 $\quad$ (4) How?

hence, $\quad a = \frac{1}{3}$ or -2. $\quad$ (5) How?

Check the work by substituting each of these values in (1).

EXERCISES

Solve each of the following and check:

1. $t^2 - 3t = 4$
2. $n^2 - n = 6$
3. $g^2 - 5g = 6$
4. $4r^2 - 3r = 7$
5. $s^2 - 5s = -4$
6. $m^2 - 7m = 3\frac{3}{4}$
7. $25r^2 - 5r = 6$
8. $9k^2 - 4k = \frac{5}{9}$
9. $4h^2 - 3h = 1$
10. $2p^2 - 3p = 2$
11. $(a + 2)(a + 3) + a^2 + 3 = (a + 3)^2 - (a - 1)^2$
12. $(2t - 3)(2t + 3) + 14 = (t + 7)^2 - t^2$
13. The area of a circle is $19\frac{9}{14}$ sq. in. Find its radius, using π as $\frac{22}{7}$.
14. The area of a rectangle is 5 sq. ft. If the length is $\frac{1}{2}$ ft. greater than the width, what are its dimensions?

148. Factorable Indices of Roots.—As $(a^{\frac{1}{3}})^{\frac{1}{2}} = a^{\frac{1}{6}}$, hence, by reversing the process, $m^{\frac{1}{6}} = (m^{\frac{1}{3}})^{\frac{1}{2}}$ and $8^{\frac{1}{6}} = (8^{\frac{1}{3}})^{\frac{1}{2}} = 2^{\frac{1}{2}}$.

Whenever possible, factor the index of the root so that the root for one factor of the index can be found. Then find this *possible root,* as was done for $8^{\frac{1}{6}}$.

EXERCISES

Simplify each of the following by the above process whenever possible:

1. $4^{\frac{1}{6}}$	8. $27^{\frac{1}{9}}$	15. $8^{\frac{1}{6}}$	22. $(4a^2b^4)^{\frac{1}{4}}$
2. $9^{\frac{1}{6}}$	9. $49^{\frac{1}{6}}$	16. $216^{\frac{1}{6}}$	23. $(25m^4n^2)^{\frac{1}{6}}$
3. $125^{\frac{1}{6}}$	10. $25^{\frac{1}{6}}$	17. $125^{\frac{1}{9}}$	24. $(125m^3n^9)^{\frac{1}{6}}$
4. $36^{\frac{1}{4}}$	11. $4^{\frac{1}{4}}$	18. $64^{\frac{1}{4}}$	25. $(100a^2b^2)^{\frac{1}{8}}$
5. $49^{\frac{1}{4}}$	12. $32^{\frac{1}{10}}$	19. $144^{\frac{1}{4}}$	26. $(16a^4m^4)^{\frac{1}{4}}$
6. $16^{\frac{1}{6}}$	13. $100^{\frac{1}{4}}$	20. $144^{\frac{1}{6}}$	27. $(4a^2b^2)^{\frac{1}{8}}$
7. $16^{\frac{1}{8}}$	14. $100^{\frac{1}{6}}$	21. $64^{\frac{1}{12}}$	28. $(36a^4c^2)^{\frac{1}{6}}$

29. The velocity of a falling body is expressed in feet per second by

$$v^2 = 64s,$$

s being the distance in feet that the body has fallen. Solve the equation for v.

30. With what velocity does a stone dropped from a height of 50 ft. strike the ground?

31. Solve the equation in Ex. 29 for s. From what height must a stone be dropped to strike the ground with a velocity of 16 ft. per second? of 60 ft. per second?

32. Find the side of the square whose area is $\frac{25}{36}$ sq. ft.; $\frac{16}{49}$ m.^{2}; $\frac{81}{25}$ dm.^{2}

33. Measure the length and width of your schoolroom. Compute the length of a diagonal along the floor. See Ex. 11, page 154.

34. The volume of a sphere in terms of its surface is

$$V = \frac{S^{\frac{3}{2}}}{6\pi^{\frac{1}{2}}}.$$

Find V when S is 8 sq. ft. Use $\frac{22}{7}$ for π.

149. Root of a Factor of a Number.—According to the meaning of exponents $(a^4b^6)^{\frac{1}{2}} = a^2b^3$. How? If the square root of each factor is first found and the roots multiplied together, the result will be the same as that found above. Thus,

$$(a^4)^{\frac{1}{2}} \times (b^6)^{\frac{1}{2}} = a^2 \times b^3 = a^2b^3.$$

Similarly, $100^{\frac{1}{2}} = (4 \times 25)^{\frac{1}{2}} = (4)^{\frac{1}{2}} \times (25)^{\frac{1}{2}} = 2 \times 5 = 10$, which is also the square root of 100. In simplifying radicals we look for a factor of the number of which the required root can be taken. Thus,

$$(75)^{\frac{1}{2}} = (25 \times 3)^{\frac{1}{2}} = 5(3)^{\frac{1}{2}}.$$

It is now only necessary to find $(3)^{\frac{1}{2}}$ and multiply this by 5.

EXERCISES

In the following, look for factors of which the required root can be taken and place the result in the form $5(3)^{\frac{1}{2}}$:

1. $12^{\frac{1}{2}}$	7. $48^{\frac{1}{2}}$	13. $32^{\frac{1}{4}}$	19. $216^{\frac{1}{2}}$
2. $50^{\frac{1}{2}}$	8. $63^{\frac{1}{2}}$	14. $56^{\frac{1}{2}}$	20. $147^{\frac{1}{2}}$
3. $8^{\frac{1}{2}}$	9. $16^{\frac{1}{3}}$	15. $81^{\frac{1}{3}}$	21. $144^{\frac{1}{4}}$
4. $18^{\frac{1}{2}}$	10. $54^{\frac{1}{3}}$	16. $125^{\frac{1}{2}}$	22. $375^{\frac{1}{3}}$
5. $27^{\frac{1}{2}}$	11. $32^{\frac{1}{3}}$	17. $32^{\frac{1}{2}}$	23. $1000^{\frac{1}{2}}$
6. $28^{\frac{1}{2}}$	12. $72^{\frac{1}{2}}$	18. $48^{\frac{1}{3}}$	24. $1000^{\frac{1}{3}}$

25. Draw a square and one of its diagonals. Mark the sides a and the diagonal d. Explain why

$$d^2 = 2a^2.$$

26. Solve the equation in Ex. 25 for d.

27. Show that the diagonal of a baseball diamond is $90\sqrt{2}$ *ft.* Find this distance to two decimal places.

150. Simplifying Radicals.—Before extracting the root of any radical, reduce it by use of the principles of the preceding articles to the form which requires the fewest and the smallest roots to be taken.

EXERCISES

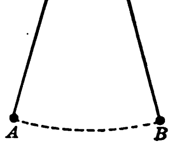
Simplify each of the following, as much as possible, and find the required root whenever you can:

1. $12^{\frac{1}{2}}$
2. $44^{\frac{1}{2}}$
3. $(\frac{8}{5})^{\frac{1}{2}}$
4. $28^{\frac{1}{2}}$
5. $(\frac{3}{4})^{\frac{1}{2}}$
6. $300^{\frac{1}{2}}$
7. $(\frac{5}{8})^{\frac{1}{2}}$
8. $(\frac{1}{2})^{\frac{1}{2}}$
9. $25^{\frac{1}{4}}$
10. $49^{\frac{1}{4}}$
11. $64^{\frac{1}{4}}$
12. $98^{\frac{1}{2}}$
13. $72^{\frac{1}{2}}$
14. $(\frac{3}{4})^{\frac{1}{3}}$
15. $(\frac{8}{16})^{\frac{1}{4}}$
16. $(\frac{3}{8})^{\frac{1}{4}}$
17. $48^{\frac{1}{2}}$
18. $80^{\frac{1}{4}}$
19. $(75a^4)^{\frac{1}{2}}$
20. $(225t^6)^{\frac{1}{2}}$
21. $(\frac{36}{49})^{\frac{1}{4}}$
22. $\left(\frac{5a}{7b}\right)^{\frac{1}{2}}$
23. $288^{\frac{1}{2}}$
24. $(\frac{4}{9})^{\frac{1}{4}}$
25. $(\frac{1}{32})^{\frac{1}{5}}$
26. $\left(\frac{3a}{2b}\right)^{\frac{1}{2}}$
27. $\left(\frac{2a^2}{5b^3}\right)^{\frac{1}{2}}$
28. $(108x^2y^2)^{\frac{1}{2}}$
29. $(245a^2b^2)^{\frac{1}{2}}$
30. $(500a^4x^8)^{\frac{1}{2}}$

31. It is found in physics that the time, t, in seconds, which it takes a pendulum to make one swing from AO to BO is

$$t = \pi\sqrt{\frac{l}{32}}$$

in which l is given in feet. Find t when l is 8 ft.; 16 ft.; 4 ft.



32. Square both members of the equation in Ex. 31. From this solve for l. What does l state?

33. Find the length of the pendulum which makes one swing from OA to OB in 1 sec.; in 4 sec.

151. Use of Tables of Roots.—It was found that $(12)^{\frac{1}{2}} = 2(3)^{\frac{1}{2}}$. On page 232 will be found the square and the cube roots of numbers. All that is now necessary to find $(12)^{\frac{1}{2}}$ is to look up the square root of 3 and multiply it by 2. Similarly,

$$(\tfrac{2}{3})^{\frac{1}{2}} = (\tfrac{2}{3} \times \tfrac{3}{3})^{\frac{1}{2}} = (\tfrac{6}{9})^{\frac{1}{2}} = \frac{(6)^{\frac{1}{2}}}{3},$$

so that it is only necessary to find the square root of 6 and divide by 3.

By means of the tables, find the square roots and the cube roots asked for in the exercises on page 158.

Hold a number contest on simplifying radicals and finding roots by use of the tables.

152. Tables in Solving Quadratic Equations.—In the solutions of quadratic equations radicals often arise. These can be simplified by the use of the tables of roots. Thus,

$$3z^2 - 2 = 0 \qquad (1)$$
$$3z^2 = 2 \qquad (2) \text{ How?}$$
$$z^2 = \tfrac{2}{3} \qquad (3) \text{ How?}$$
$$z = (\tfrac{2}{3})^{\frac{1}{2}} \qquad (4) \text{ How?}$$
$$= \pm \tfrac{1}{3}(6)^{\frac{1}{2}}. \qquad (5) \text{ How?}$$

From the tables we find that $(6)^{\frac{1}{2}}$ is 2.449,

hence, $z = \pm \frac{1}{3}(2.449) = \pm .816.$ (6) How?

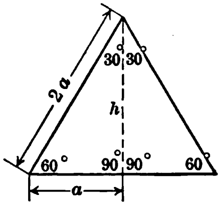
Check these results by substituting each in (1).

EXERCISES

Solve Exs. 1–8 by the use of the tables and check:

1. $2a^2 + 3 = 8$
2. $5m^2 - 3 = 3m^2$
3. $4r^2 - 1 = 7$
4. $3p^2 + 2 = 9$
5. $g^2 + 4g = 1$
6. $s^2 + 6s = 2$
7. $4h^2 - 12h = 5$
8. $9w^2 - 30w = 2$

9. Draw an equilateral triangle. Show that the altitude, h, bisects the vertical angle and the base. From the right triangle



$$(2a)^2 = a^2 + h^2.$$

Find the value of h in terms of a.

10. Find the altitude of an equilateral triangle whose sides are each 6 in.; 10 cm.; 3 ft.

11. Find the area of each triangle in Ex. 10.

12. In finding distances surveyors use the formula:

$$c^2 = a^2 + b^2 - 2ab.$$

Suppose $c = 5$ and $b = 3$. Substitute in the formula and find the value of a.

13. The following is a formula used in measuring lenses:

$$R = \frac{l^2}{6d} + \frac{d}{2}.$$

Multiply by the quantity which will leave an expression without fractions.

14. Find the value of d, if $R = 5$ and $l = 3$.

15. Solve Exs. 8, 9, 10, 18, and 24 on page 113.

16. The diameter, D, in inches, of the steam-pipe needed for an engine of horse-power $H.P.$ is

$$D = \sqrt{\frac{H.P.}{6}}.$$

Find the diameter of the steam-pipe for an engine of 24 horse-power; of 30 horse-power; of 40 horse-power.

Hold a number contest on extracting roots.

153. Quadratic Equations Solved by Formula.—Suppose the following equation is solved for x, in which a, b, and c stand for general numerical values.

$$ax^2 + bx + c = 0 \quad (1)$$

$$ax^2 + bx = -c \quad (2) \quad \text{How?}$$

$$a^2x^2 + abx = -ac, \quad (3) \quad \text{How?}$$

$\frac{b^2}{4}$ is now added to both members so as to make the left-hand member a perfect square. How is $\frac{b^2}{4}$ obtained?

$$a^2x^2 + abx + \frac{b^2}{4} = \frac{b^2}{4} - ac \quad (4) \quad \text{How?}$$

$$= \frac{b^2 - 4ac}{4}. \quad (5)$$

Hence,
$$ax + \frac{b}{2} = \frac{\pm\sqrt{b^2 - 4ac}}{2} \quad (6) \quad \text{How?}$$

$$ax = \frac{-b \pm \sqrt{b^2 - 4ac}}{2} \quad (7) \quad \text{How?}$$

and
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \quad (8) \quad \text{How?}$$

After a quadratic equation has been reduced to form (1), the two values of the unknown can be found by substituting the values of a, b, and c in (8). Thus, in

$$2t^2 - 8t + 3 = 0, \quad (9)$$

$a = 2$, $b = -8$, $c = 3$. Hence, substituting in (8) gives

$$t = \frac{-(-8) \pm \sqrt{8^2 - 4 \times 2 \times 3}}{4} = \frac{8 \pm \sqrt{64 - 24}}{4} \quad (10)$$

$$= \frac{8 \pm \sqrt{40}}{4} = \frac{8 \pm 2\sqrt{10}}{4} = \frac{4 \pm \sqrt{10}}{2} \quad (11)$$

$$= \frac{4 \pm 3.162}{2} = 3.58 \text{ or } .419. \quad (12) \quad \text{How?}$$

Check by substituting each of these values in (9).

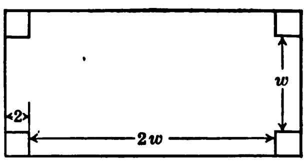
Equation (8) may therefore be used as a **formula** for the *solution* of quadratic equations.

EXERCISES

In solving the following, first reduce each to the form of equation (1) on page 162, and then use equation (8) as a formula. Use table of square roots whenever possible. Solve the first five also by factoring and compare results.

1.	$3r^2 + 5r + 2 = 0$	10.	$-3z^2 + 2z = -7$
2.	$t^2 + 2 = -3t$	11.	$(a+2)^2 = 7 + a$
3.	$t^2 + t - 6 = 0$	12.	$3p^2 - 4p = k$ Find p.
4.	$6r^2 - r = 2$	13.	$gt^2 + 2gt + k = 0$ Find t.
5.	$r^2 - 4r - 5 = 0$	14.	$mn^2 + qn = r$ Find n.
6.	$x^2 - 8x + 3 = 0$	15.	$m^2 + m - 3 = 8$
7.	$2t^2 = 2 - 3t$	16.	$(a+3)(a-2) + 3 = 0$
8.	$2y^2 - 8y + 7 = 0$	17.	$ka^2 - pa - q = 0$ Find a.
9.	$\frac{5a^2}{3} - \frac{a}{6} = 2$	18.	$\frac{3t^2}{2} - t - \frac{1}{4} = 0$

19. The distance from one corner of a room to the opposite corner is 24 ft. What are the dimensions of the floor if the length is 6 ft. more than the width?



20. An open box was made by cutting a 2-in. square out of the corners of a piece of pasteboard. The box is twice as long as it is wide and the surface contains 40 sq. in. What are the dimensions of the box?

21. A piece of zinc is 10 in. longer than it is wide. It has cut in it a square hole whose area is 18 sq. in. The remaining area is 10 sq. in. What are the dimensions of the zinc?

154. Addition and Subtraction.—Place like terms under each other and then add or subtract, as the case may be. Numerical radicals can also be combined, as,

$$\sqrt{3} + 5\sqrt{2} - 4\sqrt{3} + \sqrt{2} = 6\sqrt{2} - 3\sqrt{3}. \qquad \text{How?}$$

$$5(3)^{\frac{1}{2}} + 4(7)^{\frac{1}{2}} - (3)^{\frac{1}{2}} + 8(7)^{\frac{1}{2}} = 4(3)^{\frac{1}{2}} + 12(7)^{\frac{1}{2}}. \qquad \text{How?}$$

It is also possible to change numbers with fractional exponents so that they may be combined. Thus,

$$(2)^{\frac{1}{2}} + (18)^{\frac{1}{2}} + (50)^{\frac{1}{2}} = (2)^{\frac{1}{2}} + 3(2)^{\frac{1}{2}} + 5(2)^{\frac{1}{2}} = 9(2)^{\frac{1}{2}}.$$

EXERCISES

1. Add: $5a^{\frac{1}{2}} + 3b^{\frac{1}{2}} - 2c^{\frac{1}{2}}$; $3a^{\frac{1}{2}} + 4c^{\frac{1}{2}} + 7b^{\frac{1}{2}}$; $-2a^{\frac{1}{2}} + 9c^{\frac{1}{2}}$.
2. Add: $3m^{\frac{1}{2}}n^{\frac{1}{3}} + 7m^2n^{\frac{1}{2}} - 2b^{\frac{1}{4}}$; $2m^{\frac{1}{2}}n^{\frac{1}{3}} - 5m^2n^{\frac{1}{2}} + 3b^{\frac{1}{4}}$; $5b^{\frac{1}{4}} - 2m^2n^{\frac{1}{2}} + 4m^{\frac{1}{2}}n^{\frac{1}{3}}$; $2m^2n^{\frac{1}{2}} + 3m^{\frac{1}{2}}n^{\frac{1}{3}}$.
3. Simplify, $2\sqrt{3} + 3\sqrt{2} - 5\sqrt{3} - 3\sqrt{2} + 9\sqrt{3}$.
4. Simplify, $\sqrt{3ab} + 2\sqrt{cd} - 3\sqrt{3ab} + 2\sqrt{cd} - 4\sqrt{3ab}$.
5. Simplify, $3\sqrt{a} + 7\sqrt{b} - (2\sqrt{a} - 9\sqrt{b}) + 2\sqrt{a}$.
6. From $5\sqrt{x} + 3\sqrt{y}$ take $2\sqrt{x} - 7\sqrt{y}$.
7. From $7\sqrt{ab} + 3\sqrt{cd} - 7\sqrt{m}$ take $2\sqrt{ab} - 3\sqrt{cd} + 2\sqrt{m}$.
8. From $3a^{\frac{1}{2}} - ab^{\frac{1}{3}} + b^{\frac{1}{4}}$ take $5a^{\frac{1}{2}} - 7ab^{\frac{1}{3}} - 9b^{\frac{1}{4}}$.
9. From $2xy^{\frac{1}{5}} - 3x^2y^{\frac{1}{4}} + x^{\frac{1}{3}}y^2$ take $xy^{\frac{1}{5}} + 7x^2y^{\frac{1}{4}} - 4x^{\frac{1}{3}}y^2$.
10. From $8\sqrt{a+b} + 3\sqrt{c+d}$ take $3\sqrt{a+b} + 9\sqrt{c+d}$.

Simplify the radicals and combine when possible:

11. $3^{\frac{1}{2}} + 12^{\frac{1}{2}} + 75^{\frac{1}{3}}$
12. $5^{\frac{1}{2}} + 20^{\frac{1}{2}} - 80^{\frac{1}{2}}$
13. $72^{\frac{1}{2}} + 98^{\frac{1}{2}} - 3(2)^{\frac{1}{2}}$
14. $45^{\frac{1}{2}} + 5^{\frac{1}{2}} + 20^{\frac{1}{2}}$
15. $125^{\frac{1}{2}} - 3(5)^{\frac{1}{2}}$
16. $50^{\frac{1}{2}} + 162^{\frac{1}{2}} - 8^{\frac{1}{2}}$

17. Find the approximate value of the result in Ex. 12 by use of the tables.

18. Find the approximate values of the results in Exs. *14 and 16.*

155. Multiplication and Division.—The laws laid down for multiplication and division of literal numbers with whole numbers as exponents hold in all cases of numbers having fractional exponents. Thus,

$$\begin{array}{l}
3w^{\frac{2}{3}} + 7w^{\frac{1}{3}}v^{\frac{1}{3}} - 5v^{\frac{2}{3}} \\
\qquad\quad 2w^{\frac{1}{3}} + 4v^{\frac{1}{3}} \\
\hline
6w + 14w^{\frac{2}{3}}v^{\frac{1}{3}} - 10w^{\frac{1}{3}}v^{\frac{2}{3}} \\
\qquad\quad 12w^{\frac{2}{3}}v^{\frac{1}{3}} + 28w^{\frac{1}{3}}v^{\frac{2}{3}} - 20v \\
\hline
6w + 26w^{\frac{2}{3}}v^{\frac{1}{3}} + 18w^{\frac{1}{3}}v^{\frac{2}{3}} - 20v
\end{array}$$

Similarly for division,

$$\begin{array}{r|l}
 & 9m - 3m^{\frac{1}{2}}n^{\frac{1}{2}} + n \\
\hline
3m^{\frac{1}{2}} + n^{\frac{1}{2}} & 27m^{\frac{3}{2}} \qquad\qquad + n^{\frac{3}{2}} \\
 & 27m^{\frac{3}{2}} + 9mn^{\frac{1}{2}} \\
\cline{2-2}
 & \qquad - 9mn^{\frac{1}{2}} \\
 & \qquad - 9mn^{\frac{1}{2}} - 3m^{\frac{1}{2}}n \\
\cline{2-2}
 & \qquad\qquad\qquad 3m^{\frac{1}{2}}n + n^{\frac{3}{2}} \\
 & \qquad\qquad\qquad 3m^{\frac{1}{2}}n + n^{\frac{3}{2}} \\
\cline{2-2}
\end{array}$$

EXERCISES

1. Explain each of the above operations fully.

Carry out the following operations as indicated:

2. $(a^{\frac{1}{2}} - b^{\frac{1}{2}})(a^{\frac{1}{2}} + b^{\frac{1}{2}})$
3. $(a^{\frac{2}{3}} + b^{\frac{2}{3}})^2$
4. $(5a^{\frac{1}{2}} - b^{\frac{1}{2}})(5a^{\frac{1}{2}} + 3b^{\frac{1}{2}})$
5. $(3t^{\frac{2}{3}} + 5t^{\frac{1}{3}})(3t^{\frac{2}{3}} - 5t^{\frac{1}{3}})$
6. $(12g^{\frac{1}{2}}h^{\frac{1}{2}} - 5g^{\frac{1}{3}}h^{\frac{1}{2}})^2$
7. $(a^{\frac{1}{2}}b^{\frac{1}{2}} - 7c^{\frac{1}{2}})(a^{\frac{1}{2}}b^{\frac{1}{2}} + 3c^{\frac{1}{2}})$
8. $(3r^{\frac{1}{2}}s^{\frac{1}{3}}t^{\frac{3}{4}} - 9)(3r^{\frac{1}{2}}s^{\frac{1}{3}}t^{\frac{3}{4}} + 3)$
9. $(a - b) \div (a^{\frac{1}{2}} - b^{\frac{1}{2}})$
10. $(t^3 - u^3) \div (t^{\frac{3}{2}} + u^{\frac{3}{2}})$
11. $(a + 2a^{\frac{1}{2}}b^{\frac{1}{2}} + b) \div (a^{\frac{1}{2}} + b^{\frac{1}{2}})$
12. $(x - 2x^{\frac{1}{2}}y^{\frac{1}{2}} + y) \div (x^{\frac{1}{2}} - y^{\frac{1}{2}})$
13. $(z - 7z^{\frac{1}{2}} + 12) \div (z^{\frac{1}{2}} - 3)$
14. $(t^{\frac{1}{2}} - 8t^{\frac{1}{4}} + 16) \div (t^{\frac{1}{4}} - 4)$
15. $(9r - 15r^{\frac{1}{2}} - 14) \div (3r^{\frac{1}{2}} - 7)$
16. Multiply $5a^{\frac{3}{4}} - 2a^{\frac{1}{2}}c^{\frac{1}{2}} + c^{\frac{3}{4}}$ by $2a^{\frac{1}{4}} - 3c^{\frac{1}{4}}$
17. Divide $10x - x^{\frac{2}{3}}y^{\frac{1}{3}} - 13x^{\frac{1}{3}}y^{\frac{2}{3}} - 12y$ by $5x^{\frac{2}{3}} + 7x^{\frac{1}{3}}y^{\frac{1}{3}} + 4y^{\frac{2}{3}}$

156. Multiplication and Division with Radicals.—In multiplication and division first express any radical in the exponential form. Thus,

$(\sqrt{3} + 5\sqrt{7}) \times (2\sqrt{3} + 3\sqrt{7})$ becomes $[(3)^{\frac{1}{2}} + 5(7)^{\frac{1}{2}}] \times [2(3)^{\frac{1}{2}} - 3(7)^{\frac{1}{2}}]$.

In carrying out the multiplication note that $(3)^{\frac{1}{2}} \times (3)^{\frac{1}{2}} = 3^1 = 3$ and that $(3)^{\frac{1}{2}} \times (7)^{\frac{1}{2}} = (3 \times 7)^{\frac{1}{2}} = (21)^{\frac{1}{2}}$.

$$\begin{array}{rr} (3)^{\frac{1}{2}} & + 5(7)^{\frac{1}{2}} \\ 2(3)^{\frac{1}{2}} & - 3(7)^{\frac{1}{2}} \\ \hline 6 & + 10(21)^{\frac{1}{2}} \\ -105 & - 3(21)^{\frac{1}{2}} \\ \hline -99 & + 7(21)^{\frac{1}{2}} \end{array}$$

EXERCISES

Perform the indicated operations:

1. $(\sqrt{2} + \sqrt{3})^2$
2. $(3\sqrt{5} + 5\sqrt{3})^2$
3. $(4\sqrt{3} - 5\sqrt{2})^2$
4. $(\sqrt{2} + \sqrt{5})(\sqrt{3} + \sqrt{5})$
5. $(2\sqrt{3} + 5)(2\sqrt{3} - 5)$
6. $(3\sqrt{7} - \sqrt{2})(4\sqrt{7} + \sqrt{2})$
7. $(\sqrt{x} + 5 + 7)^2$
8. $(\sqrt{a + b} + 5)^2$
9. $(\sqrt{r + t} - 6)^2$
10. $(\sqrt{7} - 2\sqrt{5})(3\sqrt{7} + 4\sqrt{5})$

Hold a number contest on operations with radicals.

157. Radical Equations.—Equations often arise containing radicals or the unknown raised to a power which is a fraction.

I.

$$\begin{array}{rcll} z &=& 2 + \sqrt{z^2 - 8} & (1) \\ z - 2 &=& \sqrt{z^2 - 8} & (2) \text{ How?} \\ z^2 - 4z + 4 &=& z^2 - 8 & (3) \text{ How?} \\ -4z &=& -12 & (4) \text{ How?} \\ z &=& 3 & (5) \text{ How?} \end{array}$$

Check in (1).

II.

$$\begin{array}{rcll} \sqrt[3]{a + 2} &=& 3 & (1) \\ a + 2 &=& 27 & (2) \text{ How?} \\ a &=& 25 & (3) \text{ How?} \end{array}$$

Check in (1).

III.

$$a^{\frac{3}{2}} - 3 = 5 \quad (1)$$
$$a^{\frac{3}{2}} = 8 \quad (2) \text{ How?}$$
$$(a^{\frac{3}{2}})^{\frac{2}{3}} = (8)^{\frac{2}{3}} \quad (3) \text{ How?}$$
$$a = 4 \quad (4) \text{ How?}$$

Check in (1).

IV.

$$a^{\frac{1}{2}} + 4a^{\frac{1}{4}} = 5 \quad (1)$$
$$a^{\frac{1}{2}} + 4a^{\frac{1}{4}} + 4 = 9 \quad (2) \text{ How?}$$
$$a^{\frac{1}{4}} + 2 = \pm 3 \quad (3) \text{ How?}$$
$$a^{\frac{1}{4}} = \pm 3 - 2 = 1 \text{ or } -5 \quad (4) \text{ How?}$$
$$a = 1 \text{ or } 625 \quad (5) \text{ How?}$$

Check in (1).

Note in checking in IV that $(625)^{\frac{1}{2}} = \pm 25$. We use $+ 25$, since we must find $a^{\frac{1}{4}}$ and $(-25)^{\frac{1}{2}}$ has no meaning. $(625)^{\frac{1}{4}} = [(625)^{\frac{1}{2}}]^{\frac{1}{2}} = [25]^{\frac{1}{2}} = \pm 5$. If we use $a^{\frac{1}{4}} = -5$, then (1) becomes

$$25 + 4(-5) = 5$$
$$25 - 20 = 5.$$

If we use $a^{\frac{1}{4}} = +5$, then (1) becomes $25 + 4(5)$, which does **not** equal 5. Hence -5 is a root but **not** $+5$.

EXERCISES

Find the numerical value of the literal quantity in each of the following twenty-eight equations and check the result:

1. $x^{\frac{1}{2}} = 3$
2. $\sqrt{y + 1} = 2$
3. $\sqrt{2a + 3} = 5$
4. $\sqrt{t^2 + 2} = 1 + t$
5. $\sqrt{a + 2} + 3 = 4$
6. $\sqrt{2r + 7} + 2 = 5$
7. $\sqrt{2r + 1} = \sqrt{r + 4}$
8. $(a + 1)^{\frac{1}{2}} = 3$
9. $t = 5 - \sqrt{t^2 + 5}$
10. $\sqrt{t^2 + 3} = 1 + t$
11. $\sqrt{x + 1} + \sqrt{x - 1} = 2$
12. $6 - t^{\frac{1}{2}} = (t + 12)^{\frac{1}{2}}$
13. $\sqrt{y + 1} = 2 + \sqrt{y - 7}$
14. $t^{\frac{3}{2}} + 2 = 10$
15. $\sqrt{a^2 - 2a + 8} = a - 4$
16. $\sqrt{c^2 + 7c - 4} = \sqrt{c^2 + 5c - 1}$
17. $s - 8s^{\frac{1}{2}} + 15 = 0$
18. $2b + 7b^{\frac{1}{2}} - 4 = 0$

19. $2a = (3a + 1)^{\frac{1}{2}}$

20. $\sqrt{5a^2 + 4} = a + 4$

21. $\sqrt{t^2 + 10} - t = 4$

22. $\sqrt[3]{a^2 + 2a + 12} = 3$

23. $\dfrac{x + 7}{x^{\frac{1}{2}}} = x^{\frac{1}{2}} + 1$

24. $r^{\frac{3}{2}} + 2r^{\frac{3}{4}} - 3 = 0$

25. $m^{\frac{1}{2}} - 4m^{\frac{1}{4}} + 4 = 0$

26. $\sqrt{a + 20} = 3$

27. $\dfrac{3}{\sqrt{a + 4}} = \sqrt{a - 4}$

28. The time in seconds, t, that it takes a pendulum l ft. long to make one swing is given by the equation,

$$t = \pi\sqrt{\frac{l}{32}}.$$

Solve the equation for l. What is the length when the time of swing is 1 sec.? 2 sec.? 3 sec.?

29. The velocity, v, of a body falling from rest for a distance, d, is given by the equation,

$$v = \sqrt{64d}.$$

Find the value of d. From what height must a body have fallen to have gained a velocity of 8 ft. per second? 32 ft. per second? 64 ft. per second?

30. The volume in gallons of a cylindrical tank with height in inches, h, and diameter in inches, d, is

$$v = .0034hd^2.$$

Solve for d.

31. What is the diameter of the tank that contains 600 gal. and has an altitude of 3 ft.? 1800 gal. and altitude 4 ft. 6 in.?

32. The volume of a sphere is

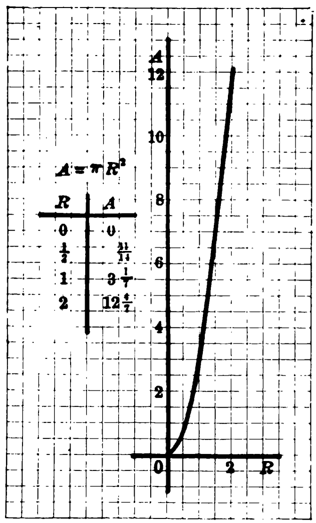
$$V = \tfrac{4}{3}\pi R^3.$$

Solve for R. Find R when V is 4500.

Hold a number contest on operations with radicals.

IX

EXTENSION OF GRAPHS



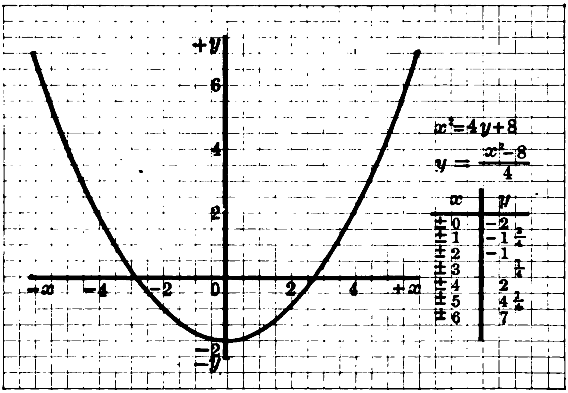
158. Graphs of Quadratic Equations.—In Art. 129 we graphed equations having variables both of the first degree, which gave straight lines. If the exponent of one or both of the variables is greater than 1, the graph of the equation will not be a straight line. In the above equation, why is it simpler to take R as the independent variable?

EXERCISES

1. Check the graph above for several points.

2. From the graph find the radius of the circle whose area is 8 sq. in.; 9.5 cm.^{2}; 6 cd.^{2}; 8.5 sq. ft.; 10.25 m.^{2}.

3. From the graph find the area of the circle whose radius is 1.5 in.; 2 cm.; 1.25 ft.; 1.75 in.; .75 dm.



159. Graphing Quadratic Equations.—In order to graph an equation, as,

$$x^2 = 4y + 8, \qquad (1)$$

it is first solved for the variable with the lowest exponent. This gives, for (1),

$$y = \frac{x^2 - 8}{4}. \qquad (2)$$

Values are now assigned to x and the corresponding values of y found from (2). Negative as well as positive values of x must be included. The table of equivalents will then be that given to the right of the graph above. How is the graph checked?

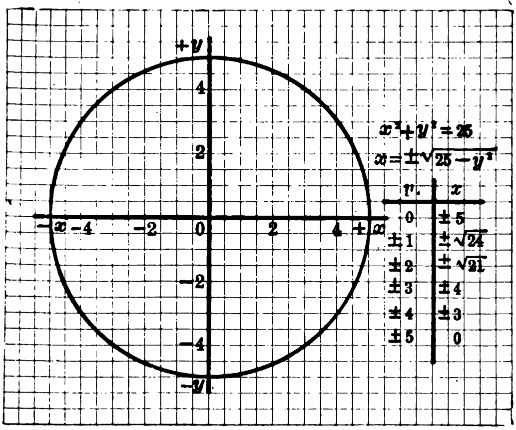
If an equation contains both variables raised to the second power, as,

$$x^2 + y^2 = 25, \qquad (3)$$

the equation is first solved for either variable, as,

$$x = (25 - y^2)^{\frac{1}{2}}. \qquad (4)$$

Values are now selected for the variable within the parentheses; that is, for y here.



Note that the number within the parentheses must be positive, as no number can be found which multiplied by itself gives a negative number. For this reason y cannot be greater than 5. Why? In computing the values of x make use of the square-root tables on page 232.

EXERCISES

1. Check the graph for equation (1), page 170.
2. Check the above graph.
3. Supply the values of x that are lacking in the table above and compute the corresponding values of y.
4. From the graph on page 170 find the value of y when x is 3; -2; -3.5; 1.25; -2.25; 0.5; -0.25.
5. From the above graph find the value of y when x is 2.5; -1.5; 0.5; -2.75; -3.25; -0.75.

Graph the following equations and check:

6. $x^2 + y^2 = 36$
7. $y^2 = 4z$
8. $g^2 = 2k - 4$
9. $4w^2 + 9v^2 = 36$
10. $m^2 - n^2 = 1$
11. $r^3 = s$

12. The space in feet, s, that a body falls from rest in any number of seconds, t, is expressed by the equation,

$$s = 16t^2.$$

In graphing the above equation place t along the horizontal axis, letting 4 spaces represent 1 sec. If you have 26 vertical spaces, let each one represent 10 ft.; otherwise use some other arrangement, so that 260 ft. can be represented for the space s.

13. Read from the graph the distance which a body will fall in the first 1.5 sec.; first 2.5 sec.; first 3.5 sec.

14. Read from the graph how long it will take a body to fall 16 ft.; 64 ft.; 50 ft.; 150 ft.; 200 ft.

15. Read from the graph how far a body falls in the 2d second; in the 3d second; in the 4th second.

16. A body dropped from rest will continually gain velocity. If it is dropped from a height, h, its velocity will be expressed by the equation,

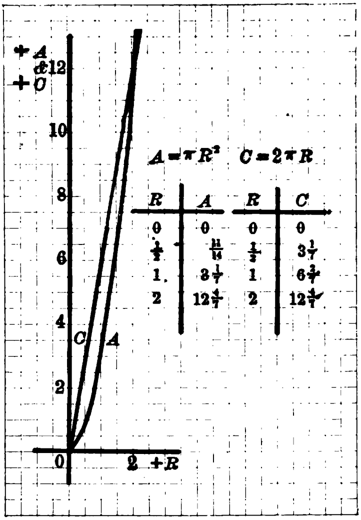
$$v^2 = 64h.$$

Graph this equation and check.

17. Read from the graph what velocity a body will attain if dropped from a height of 25 ft.; 100 ft.; 36 ft.; 62.5 ft.

18. Read from the graph from what height a body must be dropped to gain a velocity of 16 ft. per second; of 20 ft. per second; of 32 ft. per second; of 40 ft. per second.

19. Graph one of the geometrical formulas found on page 226 that contains only two variables. Use the table of square roots when helpful in finding the corresponding values of the variables. Check the work. Explain how to *read the* graph.



160. Simultaneous Quadratics.—First recall the meaning of simultaneous equations or reread pages 136 and 137.

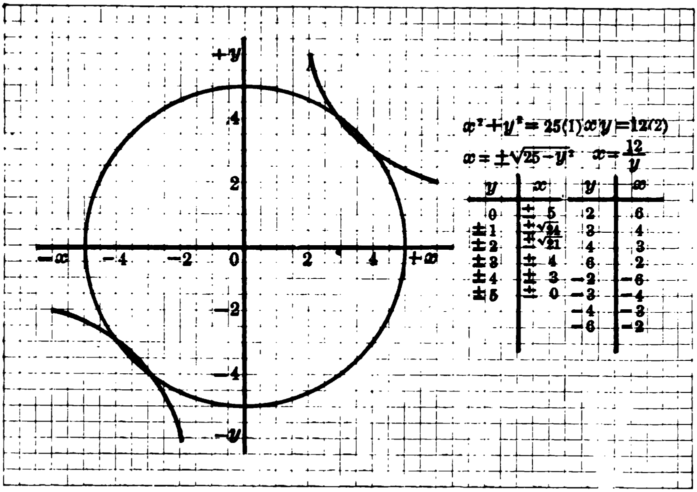
The horizontal distance gives the length of the radius for both graphs. For C the vertical axis gives the length of the circumference of the circle as in **linear feet** or **linear meters.** For A the vertical axis gives the area of the circle as in **square feet** or **square meters.** The two graphs cut each other in two points, (0, 0) and (2, $12\frac{4}{7}$), as is seen in the figure. These values which satisfy the two operations are the **simultaneous solutions** of the two equations.

EXERCISES

1. Find the values of C and A when R is .5 ft.; 1.5 in.; 2 cm.; 1.5 dm.; 1 in.; 1.25 m.

2. Find the value of A when C is 5 in.; 8.5 ft.; 12 cm.

3. Find the value of C when A is 4 sq. in.; 6 sq. ft.; 8 cm.^{2}; 12.5 m.^{2}; 8.25 sq. ft.



Equation (1) above is graphed just as the equation on page 171. In order to graph (2) it is first solved for x when

$$x = \frac{12}{y}. \qquad (3)$$

Negative as well as positive values are assigned to y to find the corresponding value of x. Care must always be taken not to omit parts of a graph, as would have been done here if negative values had not been assigned to y. Note that 0 cannot be used as one value for y, because $\frac{12}{0}$ has no meaning. A peculiar part of the graph of (2) is that it has two distinct branches.

The two graphs intersect in four points. The equations have, therefore, four simultaneous solutions. What are they? Check each set by substituting in both equations.

EXERCISES

Graph the following sets of simultaneous equations and

check the graph of each equation. Pick out the simultaneous solutions and check by substitution in the equations:

1. $m^2 + n^2 = 16$
$m + 3n = 4$

2. $y^2 = 4x$
$x + y = 3$

3. $c = d^2 + 1$
$3c + 2d = 6$

4. $x^2 = 4y$
$x = 8 - 4y$

5. $a^2 + b^2 = 17$
$a + b = 3$

6. $z + y = 8$
$zy = 7$

7. $ab = -10$
$2a + b = -4$

8. $g = k$
$g + 2k = 9$

9. $2w^2 + 3v^2 = 11$
$w + 2v = 4$

10. $x^2 - 3x + 2 = y$
$x^2 + y = 4$

11. $r^2 + 2rs + s^2 = 16$
$2r - 3s = 12$

12. $q^2 = r + \frac{1}{4}$
$2q + r = 1$

13. The sum of two numbers is 5. Represent this by an equation and graph it. The product of the two numbers is 4. State this by an equation and graph it. Pick out the simultaneous solutions and check.

14. The equations giving the distance a body falls from rest and the velocity it attains are $f = 16t^2$ and $f = 32t$. Graph both to the same set of axes, using t for the horizontal axis. What are the points of intersection? What do they tell? Check in the equations.

15. Express by an equation that the length of a rectangle is 2 ft. more than the width. Express by an equation that the area of the rectangle is 48 sq. ft. Graph each equation and find the values of length and of width satisfying both equations; that is, solve the two equations.

Hold a number contest, using whole numbers and decimals.

X

COMPARISON OF NUMBERS

161. Majorities.—The simplest way to compare two numbers is to find how much one is greater than the other. This is called the **majority** of the larger over the smaller number. Thus, the majority of 45 over 32 is 13. In an election a majority is more than one-half of all the votes. Thus, 301 is a majority of 600. What is a majority of 601?

Every four years, a little before the presidential election, each political party meets in a convention to select a presidential candidate. The successful candidate must receive a majority of the votes in the Republican convention and a two-thirds majority in the Democratic convention. Voting is continued until one candidate receives the required number of votes. Sometimes a bill or a motion must have a 60% majority or some other majority to pass. An amendment to the Constitution of the United States must be ratified by how many States before it becomes a law? Mention some amendment to the Constitution. Is one before the people at the present time?

162. Pluralities.—The candidate in an election who receives the greatest number of votes is generally declared elected. The number of votes any candidate receives above another is called his **plurality** over the other candidate. *Thus, if A, B,* and *C* receive 645, 597, and 316 votes each

respectively, A is said to have a plurality of 48 votes over B and a plurality of 329 votes over C. What is B's plurality over C?

EXERCISES

1. The Democratic National Convention in 1916 had 1093 members. How many votes did a candidate have to receive before he could be nominated?

2. The Republican National Convention in 1916 had 987 members. How many votes did a candidate have to receive before he could be nominated?

3. In an election Mr. White received 1267 votes, Mr. Baker 1193 votes, and Mr. Downs 1307 votes. Which candidate was elected? What was his plurality over each of the other two candidates?

4. How many votes would have been necessary for a majority in the election mentioned in Ex. 3?

5. The popular vote for President in 1892 was: Grover Cleveland, 5,556,918; Benjamin Harrison, 5,176,108; J. B. Weaver, 1,041,028; John Bidwell, 264,133; Samuel Wing, 21,164. What was the successful candidate's plurality over each of the other candidates?

6. What would have been a majority in Ex. 5?

7. Bring to class the data of a local election or one mentioned in your United States history. Answer the questions asked in (5) and (6).

8. In a city election requiring a two-thirds majority to carry an election to bond the city, 832 people voted. How many "yes" votes were needed for the measure to carry?

9. In an election requiring 60 % of the votes cast to be "yes," it was found that 224 of the 370 voted "yes." Did the measure carry? How many of the 224 voting "yes" could have voted "no" and the measure still have carried?

163. Mean Averages.—John weighs 55 lb., Albert 60 lb., and William 65 lb. What is their total weight? If each boy had weighed 60 lb., their total weight would still have been 180 lb. This 60 lb., the weight that each boy would have had if each had weighed the same, is called their **mean average weight,** or simply their **average** weight. A board 8 in. wide at one end and 12 in. wide at the other end will contain just as much lumber as a board of the same length and thickness but of a uniform width of 10 in. Make a drawing or use the above picture to show this fact. What use have we made of averages this year? If n is the number of quantities a, b, c, etc., then,

$$\text{Average value} = \frac{a + b + c + \text{etc.}}{n}.$$

EXERCISES

1. The outfielders of a baseball team weigh 120 lb., 118 lb., and 125 lb. Find their average weight.

2. The infielders of a baseball team weigh 103 lb., 108 lb., 102 lb., and 98 lb. Find their average weight.

3. Find the average width of a board having ends 6 in. and 10 in.; having ends 8 in. and 11 in.

4. How many safe hits did Henry average per game if he made 18 hits in 9 games? 45 hits in 15 games?

5. If he made 75 plays in the field in 15 games, how many plays did he average per game?

6. If Roy made 15 safe hits in 6 games, how many hits did he average per game? What does a fraction mean?

7. At the same rate as in Ex. 6, how many safe hits can *Roy expect* to make in 2 games? in 8 games? in 3 games?

8. Maude bought 3 books at 40 ¢, 55 ¢, and 60 ¢. Find the average price that she paid for the books.

9. Jane's grades for a certain month were 90 % in English, 80 % in mathematics, 86 % in science, 87 % in history. Find her average grade for the month.

10. John's grades for a month were 85 % in English, 90 % in mathematics, 75 % in history, 85 % in science. What was his average grade for the month?

11. The attendance for a week in a junior high school class was as follows: Monday, 32; Tuesday, 29; Wednesday, 30; Thursday, 31; Friday, 30. What was the average daily attendance for this week?

12. The 44 pupils in a certain class in a junior high school have invested $ 1456 in Liberty Loan Bonds and Thrift Stamps. What is the average investment per pupil?

13. Another class of 67 pupils in the same school has invested $ 1854. What is the average per pupil? Compare the average for this class with that for the class in Ex. 12.

14. The thermometer readings at noon for a certain week were 95°; 97°; 100°; 99°; 101°; 100°; 92°. What was the average temperature at noon for the week?

15. Ruth sold 5 bunches of asparagus at 12 ¢ each, 13 bunches at 9 ¢ each, and 7 bunches at 6 ¢ each. How many bunches did she sell? How much did she receive for all of them? What was the average price?

16. During the year Sara bought 3 books at 90 ¢ each, 1 book at $ 1.20, and 2 books at $ 1.05. What was the average price that she paid for the books?

17. Find some other averages.

Hold a number contest, using averages and common fractions.

164. Ratios.—Numbers may also be compared by finding how many times one number is another number. Thus, 12 is how many times 4? or 5 is how many times 15? Such comparisons are called **ratios**. A common way to ask these questions would be, What is the ratio of 12 to 4? Ask the corresponding question for 5 and 15. Since ratios mean comparison by division, they are expressed as fractions. The above ratios would be $\frac{12}{4}$ and $\frac{5}{15}$. Reduce each to its simplest form.

EXERCISES

1. Express the following ratios as fractions and reduce them to their simplest forms: 35 to 5; 35 to 7; 6 to 18; 3 to 18; 48 to 18; 18 to 48; 42 to 6; 6 to 42.

2. Show that the ratio of 1 to 8 equals .125; ratio of 4 to 16 equals .25; ratio of 14 to 16 equals .875.

3. Which of the following ratios are equal: 2 to 4; 6 to 2; 4 to 10; 6 to 15; 12 to 24; 15 to 5; 14 to 35?

4. Show that the ratio of 200 % to 2 is 1.

5. Show that any two equal numbers have a ratio 1.

6. A line is cut into two parts whose ratio is 1. If one part is 7 in., how long is the other part?

7. Show that the ratio of 1 to 200 is $\frac{1}{2}$ %.

8. State three ratios each equal to 4; three each equal to $\frac{1}{2}$; three each equal to 0.4.

9. How can you change two fractions so that you know which is the larger? Is $\frac{3}{5}$ or $\frac{28}{45}$ the larger?

10. Is the ratio of 4 to 5 or the ratio of 9 to 10 the larger? Is the ratio of 3 to 14 or the ratio of 1 to 5 the larger? Is the ratio of 7 to 8 or the ratio of 55 to 64 the larger?

165. Use of Ratios.—Ratios occur frequently in everyday affairs. **All measurements are ratios.** When we say that a certain length is 6 ft. we mean that it is 6 times as long as a known length called the foot; we mean that the ratio of its length to a foot is 6.

Ratios can be found only between numbers having a common unit of measure. The ratio of 6 chairs to 3 tables has no meaning. But, if the cost of each is known, the ratio of the cost of one to the cost of the other can be found. Suppose that the chairs cost \$ 35 and that the tables cost \$ 105, then the ratio of their costs is $\frac{35}{105} = \frac{1}{3}$, or the chairs cost $\frac{1}{3}$ as much as the tables. The ratio of a yard to 4 in. is the ratio of 36 in. to 4 in., or $\frac{36}{4} = 9$.

EXERCISES

1. Find the ratio of 6 in. to a yard.

2. Find the ratio of 3 in. to a foot.

3. Find the ratio of 6 oz. to a pound.

4. Find the ratio of 5 yd. to a rod.

5. John has 6 Baby Bonds and Emma has 9. What is the ratio of their values?

6. Find the ratio of the width to the length of your schoolroom.

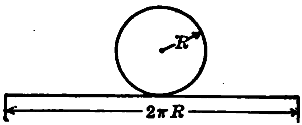
7. Find the ratio of your height to the height of the schoolroom.

8. Find the ratio of your weight to a hundredweight.

9. Find the ratio of your weight to 50 lb.

10. What is the ratio of the area of a rectangle 8 in. long by 7 in. wide to one 14 in. long by 6 in. wide?

166. Literal Ratios.—The ratio of m to n is $\frac{m}{n}$, just as the ratio of 2 to 3 is $\frac{2}{3}$. Literal ratios are of frequent occurrence. Thus, find the ratio of the circumference of a circle to its diameter. Since the circumference of a circle is πd, the ratio will be $\frac{\pi d}{d}$, which is π.



EXERCISES

1. Express in its simplest form the ratio of a^3 to a^2; the ratio of mn^2 to m^2n; the ratio of $15x^2y^3$ to $5xy^2$.

2. What is the volume of a cube with edges 1 in. long? What is the volume of a cube with edges 2 in. long? What is the ratio of the volume of the first to the volume of the second? What can you say about the volumes of two cubes if the edges of one are twice those of the other?

3. Compare in this manner the volumes of cubes, the edges of one being three times the edges of the other; also, the edges of one being five times the edges of the other.

Ratios are often expressed as three-place decimals.

4. Potatoes contain 78.3 % of water and 21.7 % of solid matter. Express as a three-place decimal the ratio of the amount of solid matter to the water in potatoes.

5. In a pound of butter there are .85 lb. of fat and in a pound of cheese there are .33 lb. of fat. Express as a three-place decimal the ratio of the amount of fat in a pound of cheese to that in a pound of butter.

6. One cow gives milk that is 3.6 % butter fat and a second cow gives milk that is 4.2 % butter fat. Find the ratio of the per cent of butter fat of the milk from the two cows.

7. A pair of shoes costing \$ 4.50 lasts 8 mo. and a pair costing \$ 6.00 lasts 12 mo. What is the ratio of the cost of the cheaper to that of the better pair? What is the ratio of the time the cheaper shoes will last to the time the better shoes will last? Which is the more economical?

8. Beef contains 19 % proteid and cheese 25 %. What is the ratio of proteid in a pound of beef to that in a pound of cheese?

9. Secure the per cents of proteids, fat, carbohydrates, and water found in several substances. Compute ratios similar to that found in Ex. 8. From current prices, ratios showing comparative values can also be found.

10. A medicine is diluted by mixing it with water in the ratio of 2 parts of medicine to 3 parts of water. Express the ratio as a fraction and as a decimal. The medicine is what per cent of the total mixture?

11. Carpenters define the pitch of a roof as the ratio of the height of the ridge above the eaves to the total span of the gable. In the picture this ratio is $\frac{AC}{BD}$. What is the pitch of the roof of a house whose ridge is 15 ft. above the eaves, the span being 20 ft.?

12. What is the pitch of a house in which AC is 8 ft. and BD is 12 ft.? in which AC is 9 ft. and BD is 15 ft.?

13. The world's railroad mileage in 1912 was 640,000 mi. Of this 242,000 mi. belonged to the United States. Express as a three-place decimal the ratio of the United States mileage to the total mileage of the world.

14. Express as a three-place decimal the ratio of the total mileage of the world to that of the United States.

167. Success Ratios.—The Giants have won 20 and lost 10 games, while the Cubs have won 10 and lost 4 games. Although the Cubs have won only one-half as many games as the Giants, they have not lost as many as the Giants. The success of a club depends upon the ratio of the games won to the total number of games played. This ratio is often referred to as the **standing,** or **average,** of the team. It is given as a three-place decimal. Thus, the average of the Giants is $\frac{20}{30} = .667$, while that of the Cubs is $\frac{10}{14} = .714$. Which team has the better average? These ratios are sometimes incorrectly referred to as per cent; as "667 per cent" and "714 per cent."

Your spelling average is the ratio of the number of words you have spelled correctly to the total number of words. Your mathematics average is the ratio of the number of problems you have solved to the total number of problems. What was this for yesterday? What is it for to-day?

EXERCISES

1. A baseball club has won 8 and lost 6 games. Find its average.

2. Alice has spelled correctly 87 words out of 90. What is her average?

3. Ray has worked correctly 19 problems out of an assignment of 20 problems. What is his average?

4. Katherine has made an average of .923 for a month in her spelling. At this rate how many words is she expected to spell correctly out of 50? out of 80? out of 350?

5. How many words is Katherine expected to miss out of 60? out of 150? out of 400?

6. James has a batting average of .216. How many safe hits will he make at this rate in 12 times at bat? in 15 times at bat? in 45 times at bat?

7. Harry's baseball team has made an average of .631 so far this season. How many of the games would they win at this rate out of the next 8 games? 15 games? 25 games?

8. The following are records of the National and the American Baseball Leagues for the season of 1915. Find the average of each club and arrange them in order for each league:

National League

	Won	Lost
Boston	83	69
Brooklyn	80	72
Chicago	73	80
Cincinnati	71	83
New York	69	83
Philadelphia	90	62
Pittsburg	73	81
St. Louis	72	81

American League

	Won	Lost
Boston	101	50
Chicago	93	61
Cleveland	57	95
Detroit	100	54
New York	69	83
Philadelphia	43	100
St. Louis	63	91
Washington	85	68

9. What is the ratio of \$ 5 to £ 1 when the English pound is worth \$ 4.866? Try to find the present value of an English pound and find the ratio for this.

10. John's father averages \$ 7.30 per day. When working during the summer John averages \$ 1.15 per day. What is the ratio of John's daily wages to his father's?

11. If you had a baseball, basket-ball, or a football team in your school the past season, find out how many games were won and how many were lost. From this compute its average. What would have been its average, if it had won 2 games more? if it had lost 2 games more?

Hold a number contest, using ratio.

168. Specific Gravity.—**Specific gravity,** or **density,** is a very common and useful **form of ratio.** The specific gravity of solids and of liquids is the ratio of the weight of the substance to the weight of an equal volume of water. Thus, since the weight of a cubic foot of cork is 15 lb. and the weight of a cubic foot of water is 62.5 lb., the specific gravity of cork is $\frac{15}{62.5} = 0.24$. Specific gravity is given as a decimal.

The specific gravity of a gas is the ratio of its weight to the weight of an equal volume of either **air** or **hydrogen,** at the same pressure as the gas. Thus, the specific gravity of coal-gas is 5 when compared with hydrogen. That is, a vessel of coal-gas will weigh 5 times as much as the same vessel filled with hydrogen. When compared with air the specific gravity of the coal-gas is 0.356.

The tables of specific gravity for use in solving the problems will be found upon page 227.

EXERCISES

1. Explain the statement that the specific gravity of coal-gas, compared with air, is 0.356.

2. What is meant by saying that the specific gravity of gold is 19.3?

3. What is the ratio of the weight of a cubic foot of gold compared with the weight of a cubic foot of water?

4. Find the weight of a cubic foot of gold.

5. Find the weight of a cubic foot of silver.

6. What does a liter of water weigh? A liter of a certain medicine weighs 1058 g. What is the specific gravity of the medicine?

7. *Find* the weight of a gallon of water.

8. Find the weight of a pint of mercury.

9. A vessel with some mercury in it weighs 23 lb. The vessel empty weighs 4.2 lb. Find the weight as well as the number of pints of mercury.

10. A cubic foot of cast iron weighs 462 lb.; find its specific gravity.

11. Find the weight of a cubic foot of ice; of a cubic yard of ice.

12. Find the weight of 1 cc. of cork; of ice; of silver.

13. Find the weight of 1 $dm.^3$ of cork; of ice; of silver.

14. Compare the weight of a plank of white pine 12 ft. long, 6 in. wide, and 4 in. thick with the weight of a plank of oak of the same dimensions.

15. What is the specific gravity of an oil, if a gallon of it weighs 7.5 lb.?

16. What is the weight of a bar of copper 14 in. long, 5 in. wide, and 0.5 in. thick?

17. How many cubic feet of cork will it take to weigh a pound? to weigh 20 lb.? to weigh 100 lb.?

18. What is the weight of the petroleum in a tank which has a square bottom 2.5 ft. on each side if filled to a depth of 7.5 ft.?

19. A petroleum tank is marked to weigh 18 lb. when empty. Its weight partly filled with petroleum is 27 lb. How many gallons of petroleum does it contain?

20. To find the specific gravity of a stone or object that sinks in water, first weigh it. Next immerse it in a glass containing some water and note how high the water rises. Weigh the glass and water. Fill the glass to where it rose with the stone immersed. Weigh again. Find the weight of the water having the volume of the stone. Find the specific gravity of the stone.

169. Unitary Analysis.—If 10 yd. of cloth cost \$1.50, what will 6 yd. cost at the same price per yard? This problem can be solved by first finding the cost of 1 yd., $\frac{1.50}{10}$, and then multiplying by 6. How much is this? As the solution is based upon finding the cost of **one** yard, it is called **unitary analysis.**

170. Proportion.—If C represents the cost of the 6 yd., the ratio of the two prices will be $\frac{C}{1.50}$. This must equal the ratio of the number of yards, $\frac{6}{10}$. Why? Hence,

$$\frac{C}{1.50} = \frac{6}{10}. \qquad (1)$$

From which,

$$C = \frac{6}{10} \times 1.50 \qquad (2) \quad \text{How?}$$

$$= .90. \qquad (3) \quad \text{How?}$$

Such a statement as (1), in which one ratio is placed equal to another, is called a **proportion.**

The above problem can be solved as well by unitary analysis as by proportion. Many problems will, however, arise in the future that can be solved simply by proportion but that are extremely difficult by any other process. This, together with the constant daily use of ratio and proportion in comparisons, makes it extremely necessary that you should know thoroughly its meaning and application.

EXERCISES

1. What is meant by unitary analysis? Make up a problem for the rest of the class to solve by unitary analysis.

2. If Harry earns \$ 1.75 in 5 da., what will he earn at *the same rate* in 9 da.? in 15 da.? in 35 da.?

3. If 10 lb. of sugar cost 95 ¢, what will 25 lb. of sugar cost?

4. If $7\frac{1}{2}$ yd. of cloth cost $ 9, what will 5 yd. of the cloth cost?

5. Ruth paid $ 16.68 for 4 Baby Bonds; what would she have to pay for 9 at the same rate? for 12? for 17?

6. Henry has earned $ 4.55 in 7 da.; how much will he earn in 5 da.? in 14 da.? in a month of 26 da.?

7. The interest on $ 500 for 2 yr. at a given rate is $ 80. What will be the interest on $ 125 for the same length of time at the same rate?

8. If 60 A. of land cost $ 3000, how many acres can be bought at the same rate for $ 2500? for $ 1500?

9. If 40 A. cost $ 1480, what will be the cost of 60 A.? of 30 A.? of 150 A.?

10. In a certain room of 28 pupils in a school the pupils had invested $ 840 in Liberty Bonds, Baby Bonds, and Thrift Stamps. In order to keep up the same rate of investment, how much should be invested by a room of 20 pupils? of 42 pupils?

11. It was found that the 32 pupils of one room in a school had $ 24 invested in Thrift Stamps. To be at the same rate, how much should be invested by the 26 pupils of another room? by the 38 pupils of yet another room?

12. By working after school and Saturdays James earns enough to buy 7 Thrift Stamps in 2 wk. At this rate how many Thrift Stamps can he buy in 9 wk.? in 6 wk.?

13. A train leaves Lake City at 9:00 a.m. and reaches Leeds, 120 mi. distant, at 1:15 p.m. How long will it take at this rate to go from Leeds to Norman, a distance of 55 mi.?

171. Proportionality; Variation.—From the equation,

$$C = \tfrac{44}{7} R, \qquad (1)$$

we find that for any circle, whatever the radius may be,

$$\frac{C}{R} = \tfrac{44}{7}. \qquad (2)$$

In other words, as the radius of a circle changes the circumference changes, but always so that the ratio of the circumference to the radius is a constant. Two numbers, as C and R, which change so that their ratio is always constant, are said to be **proportional.** Or C is said to **vary as** R. Similarly the equation,

$$A = \tfrac{22}{7} R^2, \qquad (3)$$

indicates that the area of a circle is proportional to the square of its radius. Or we say that the area of a circle varies as the square of its radius.

Note that, when two quantities are proportional or when one quantity varies as the other, **one quantity equals the other multiplied by some constant.**

One quantity may also vary as the product of two or more other quantities. For the triangle,

$$A = \tfrac{1}{2} bh. \qquad (4)$$

Hence, the area of a triangle varies as the product of its base and its altitude.

EXERCISES

Explain each of the following. Use an equation when possible.

1. The price that is paid for any number of articles is proportional to the number of articles. $P = np$.

2. The amount that is paid a person for working is proportional to the length of time that he works.

3. The amount that you are willing to pay a person for

working for you is proportional to the speed with which that person works.

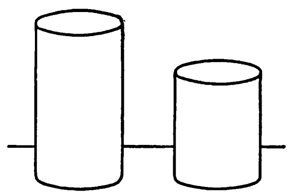
4. The rent, R, of a house for a year is proportional to the rate, r, paid per month. $R = 12r$.

5. The distance a person travels is proportional to the product of the rate and the time travelled. $D = rt$.

6. The area of a rectangle is proportional to the length of the base, if the altitude remains constant.

7. Make a statement similar to that in Ex. 6 on the change in area of a rectangle whose base remains constant and whose altitude changes.

8. The volume of a sphere is proportional to the cube of its radius.



9. The volume of a cylinder varies as its height, if the radius of the base remains constant.

10. The volume of a cone varies as the square of the radius of the base, if the altitude remains constant.

11. The interest on a certain sum of money is proportional to the time it draws interest, if the rate remains constant.

12. The interest on a certain sum of money is proportional to the rate, if the time it draws interest is constant.

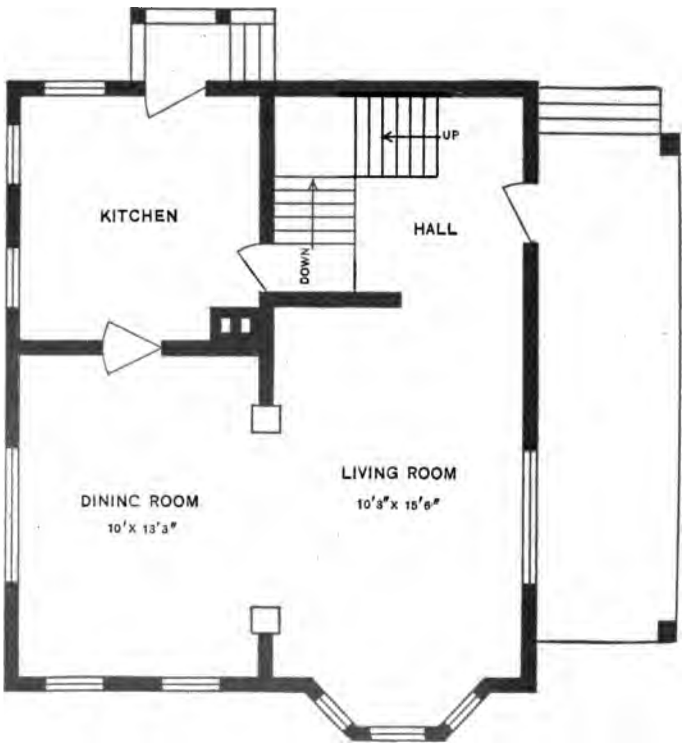
13. Interest is proportional to the product of the principal, rate, and the time.

14. The area of a sphere is proportional to what?

15. The volume of a cube is proportional to what?

16. Mary's basket-ball team has won 3 out of 5 games played. At this rate how many games would they win out of 8? out of 15? out of 14? Count a decimal over 0.5 as 1 and omit other fractions.

172. Drawing to Scale.—Draftsmen's designs, architect's plans, and all maps are figures similar to the ones that they represent. Such drawings are said to be **made to scale.** The ratio of any line in the drawing to the line that it represents is called the **scale.** Architects usually make their drawings so that $\frac{1}{4}$ in. represents 1 ft. Their scale is said to be $\frac{1}{4}$ in. to the foot, or $\frac{1}{48}$, since there are 48 quarter-inches in 1 ft. Because of a lack of space this house plan had to be drawn to a scale other than the usual architect's scale.



EXERCISES

1. Measure very accurately the drawing for the dining-room and from the dimensions marked on the plan compute the scale. Use this scale in the next two problems.

2. Find the dimensions of the porch and of the kitchen.

3. Find the width of the bow window in the living-room.

4. Plan and draw to scale a summer cottage of 3 rooms.

5. Measure your schoolroom and make a plan of it to scale.

6. The scale printed on a map is 100 mi. to the inch. What length on the map represents 50 mi.? 75 mi.? 125 mi.? 320 mi.? 45 mi.?

7. On the same map, what length is represented by .75 in.? by 2.5 in.? by 1.25 in.? by 2.45 in.?

8. The scale of a map is 250 mi. to the inch. How long a distance is represented by $\frac{1}{2}$ in.? by $\frac{3}{4}$ in.? by $1\frac{3}{8}$ in.?

9. The scale marked on a mechanical drawing is $\frac{1}{10}$. What length is represented by 4 in.? by 6.5 in.? by 3.25 in.?

10. Kansas is about 200 mi. by 400 mi. What scale would you use to draw the map upon paper 9 in. by 13 in., which will be a simple scale to work with and still give the largest possible map?

11. Look up the dimensions of your own state and answer Ex. 10 for your state.

12. Colorado is a rectangle 280 mi. by 380 mi. Draw a map of Colorado to a simple scale which will give the largest map possible on your paper.

13. Find the ratio of the volume of a box 8 in. × 15 in. × 35 in. to the volume of a box 12 in. × 52 in. × 28 in.

Hold a number contest, using proportions and the solution of equations.

173. Similar Figures.—Geometrical figures whose corresponding angles are equal and whose corresponding sides are proportional are called **similar.** Corresponding sides are also called **homologous sides.** A rectangle whose sides are 3 in. and 4 in. is similar to a rectangle whose sides are 15 in. and 20 in. Their angles are equal and $\frac{3}{15}$ equals $\frac{4}{20}$. All plans, maps, and so on just studied are similar to the figures they represent.

All circles are similar. It is proved in geometry that if two triangles have their angles equal, their corresponding sides **are proportional;** or if the corresponding sides are proportional, the angles **are equal.** Pyramids are similar if the faces and the base of one are similar to the faces and the base of the other. Cones are similar if the radius and the altitude of one are proportional to the radius and the altitude of the other. Cylinders are similar in the same way as cones. Solids bounded by planes are similar if they have similar faces similarly placed.

EXERCISES

1. Construct any triangle. Construct another triangle with sides one-half as long as the first. Measure the angles of the two triangles with your protractor. What do you find? Construct a third triangle whose sides are in some other ratio to the first triangle. Again measure the angles. What do you and the other pupils find? What is your conclusion?

2. Give three illustrations of similar figures you have seen.

3. The radius of a cone is 5 in. and its altitude 8 in. What is the radius of a cone similar to it whose altitude is 16 in.? 4 in.? 18 in.?

174. Application of Similar Figures.—Note that the triangle formed by the tree, AC, its shadow, CB, and the rays of the sun, AB, is similar to the triangle formed by the yardstick, XY, its shadow, YZ, and the rays of the sun, XZ. Hence, the ratio of the lengths of the two objects equals the ratio of the lengths of their shadows, or

$$\frac{AC}{XY} = \frac{CB}{YZ}.$$

What length is known? Which can be measured? Hence, how can the length of the tree be found?

EXERCISES

1. If $CB = 22$ ft. and $YZ = 20$ in., what is the height of the tree?

2. Find the height of a pole or a tree as the children did in the above picture.

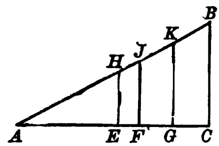
3. The sides of a flower-bed are 8 ft., 10 ft., and 12 ft. Draw this to scale using $\frac{1}{4}$ in. to the foot. Are the flower-bed and the drawing of it similar triangles?

4. If the tree above had leaned much to one side, would the value of AC found from the equation be the length of the tree? What would AC then be? How should the yardstick be held so as to find the length of the tree?

5. Draw two large right triangles that are similar but not equal.

6. In each of the triangles drawn for Ex. 5 letter the right angle C, and the acute angles A and B. Measure each side as accurately as possible. Then find to two decimal places the following ratios for each of the triangles: $\frac{AC}{AB}$; $\frac{BC}{AB}$; $\frac{AC}{BC}$; $\frac{BC}{AC}$. Compare the corresponding ratios of the two triangles. What did you and the others of the class find? What is your conclusion?

175. Ratios of Angles.—Surveyors, engineers, and scientists in general make daily application of the principles we shall study here. The ratio of any pair of sides of any of the right triangles in the figure equals the ratio of the same pair of sides in any of the other triangles. This can easily be verified by measurements, as was done in Ex. 6 above. Six of these ratios have been given definite names. The four most commonly used are:



$$\text{sine } A = \frac{\text{side opposite}}{\text{hypotenuse}} = \frac{BC}{AB};$$

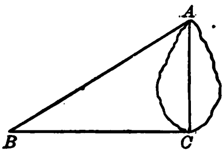
$$\text{cosine } A = \frac{\text{side adjacent}}{\text{hypotenuse}} = \frac{AC}{AB};$$

$$\text{tangent } A = \frac{\text{side opposite}}{\text{side adjacent}} = \frac{BC}{AC};$$

$$\text{cotangent } A = \frac{\text{side adjacent}}{\text{side opposite}} = \frac{AC}{BC}.$$

The abbreviations, sin A, cos A, tan A, and cot A, are always used. As the angle A increases these ratios will change; sine and tangent will increase, while cosine and cotangent will decrease. Show this. Carefully computed values of these ratios will be found on page 231. The use of these ratios is illustrated in the following problem.

In finding the distance across a pond a surveyor first drives stakes at A and C. He then lays off the line, BC, perpendicular to AC. He next measures off a distance, BC, equal to a given length and measures the angle, CBA. The right angle at C and the angle, CBA, were both measured by a transit, a picture of which is shown at the bottom of the page.



Suppose that the surveyor laid off the line, BC, 240 ft., and that he found the angle, CBA, to be 31°. Then, from the above figure,

$$\frac{AC}{BC} = \tan 31°, \text{ or } AC = BC \times \tan 31°.$$

Supplying the value of BC and of tan 31° (see page 231), we get

$$AC = 240' \times 0.6009 = \ ?$$

EXERCISES

Study carefully the explanation of the tables found on page 230 and find the following from the tables on page 231:

1. sin 40°
2. tan 21°
3. cos 8°
4. cos 15°
5. sin 19°
6. tan 47°
7. cos 56°
8. sin 73°
9. cos 81°
10. sin 35°
11. tan 40°
12. cos 13°

In making constructions calling for an angle of a given size, use the protractor, as was learned last year.

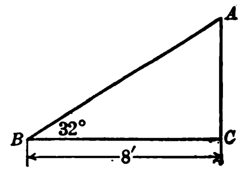
13. Draw three or four angles and measure each accurately.

14. Draw an angle of 30°; of 43°; of 76°; of 90°.

15. Draw a right triangle having one angle 60°. What is the other acute angle?

16. Measure very accurately the sides of the triangle constructed for Ex. 15. Compute the sine, cosine, tangent, and cotangent of 60°. Compare with the values in the tables. Do you see any connection between the ratios for 60° and those for 30°?

17. Construct a right triangle with one acute angle 56°. Compute the ratios asked for in Ex. 16. Compare with the values in the table.



18. Suppose that it is necessary to find the side, AC, in the accompanying figure. What ratio of the angle, CBA, is $\frac{AC}{BC}$? Since BC is 8 ft., then

$$\frac{AC}{8} = ?$$

Complete the equation and solve for AC. How can AB be found? Find AB.

19. A railroad track rises 1 ft. for every 10 ft. along the track horizontally. The per cent of rise is called the gradient of the track. What is the gradient of this track? The angle the track makes with the horizontal is called the gradient angle. Make a drawing to show the gradient angle. What is the tangent of the gradient angle? Use this to find the gradient angle from the tables. Measure the angle from your drawing to check.

20. A surveyor needing to find the distance across a river sets his transit at N. He next selects some object across the river, as at M, and *thinks* of an imaginary line, MN, connecting M to N. With his transit he finds the direction of the line, NK, making a right angle with the imaginary line, MN. A definite distance is now measured off on this line, as NK. The transit is now moved to K and the angle, NKM, measured. What ratio of the angle, NKM, is $\frac{MN}{NK}$? If we know the length of NK, how can we find the length of MN?

21. Suppose that angle, MKN, is 26° and the length of NK is 250 ft. What is the width of the river?

22. If angle MKN is 37° and the line, NK, 340 ft., what is the width of the river?

23. One of the perpendicular sides of a right triangle is 35 ft. and the angle opposite is 42°. Find the lengths of the remaining two sides. First make a drawing.

24. The boys in manual training in a junior high school made a large protractor and placed it upon a standard $5\frac{1}{4}$ ft. high. A group of the pupils of the school are shown in the picture, using the protractor to measure the height of a building. They placed the standard 27 ft. from the building. They found that they could see the top of the building along a line on the protractor making an angle of 69° with the horizontal. $\frac{MN}{HN}$ is what ratio of the angle, MHN? Hence, find the value of MN. Also find the height of the building.

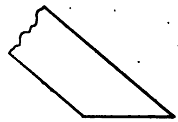
25. Four groups of the same class measured a flagpole to be 58 ft., 57 ft., 55 ft., and 59 ft. They used the average as the real height. What is this?

26. If possible, make a large protractor and place it upon a standard. Use this to measure some high objects as these children did. Take the mean of several results for the final value of the height.

Hold a number contest on finding the ratios of angles, *and finding* angles having given ratios.

27. Construct a right triangle with perpendicular sides 1.1 in. and 3.6 in. Find the tangent of the angle opposite the side 1.1 in. From the tables find this angle in degrees. Check with the protractor.

28. In order to cut the rafters for the roof of a garage at the proper angle a carpenter must know the angle the roof makes with the horizontal. The eaves are to be 14 ft. apart and the ridge is to be 6 ft. above the line joining the eaves. (See picture on page 183.) What angle are the rafters to make with the horizon?



29. Harry has played out 600 ft. of string to his kite. He estimates that the distance to the kite is 10 % less because of the slack in the string. How far would the kite then be away? If the direction to the kite makes an angle of 40° with the horizontal, how high is the kite? What is its horizontal distance away?

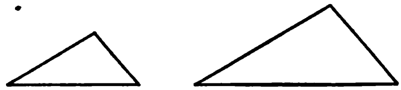
30. A stairway rises 12 ft. for every 14 ft. along the horizontal. Make a drawing to show the angle at which the banister-rail must be cut. From the tables find this angle to the nearest degree.

31. Draw any right triangle. Mark the acute angles, A and B, and the right angle, C. Express sine and cosine of angle, A, each in terms of the sides. Square each and add; that is, find $(\sin A)^2 + (\cos A)^2$. Note that the numerator and the denominator are equal. Why? Hence, what is always the value of the sum of the squares of the sine and the cosine of any angle?

32. Divide the sine of A by the cosine of A. Note that this will always be the tangent of A.

Hold a number contest on the use of the various ratios of angles.

176. Comparison of Areas and Volumes of Similar Figures.—It is proved in geometry that areas of similar plane figures are **proportional to the square** of corresponding sides or to the square of corresponding dimensions. Thus, if in two similar rectangles one is 3 times as wide as the other, their areas will vary as 9 to 1. If the bases of two similar triangles are in the ratio of $\frac{2}{5}$, their areas are in the ratio $\frac{4}{25}$. How?



It is also proved in geometry that volumes of similar solids are **proportional to the cubes** of corresponding sides or to the cubes of corresponding dimensions. Thus, if the altitudes of two similar pyramids are in the ratio $\frac{3}{4}$, their volumes will be in the ratio $\frac{27}{64}$. How?

EXERCISES

1. Compare the areas of two circles whose radii are 10 in. and 5 in.; 6 cm. and 15 cm.

2. Compare the areas of two similar triangles whose altitudes are 4 in. and 6 in.

3. The hypotenuse of a right triangle is 9 ft. and of a similar triangle is 12 in. Compare their areas.

4. One side of a pentagon is 2.4 cm. Compare its area with that of a similar pentagon whose corresponding side is 1.8 cm.; 6 cm.; 3.6 cm.

5. A triangle has an area of 30 sq. in. and an altitude of 8 in. Find the altitude of a similar triangle whose area is 48 sq. in. State first as an equation.

6. A house plan is drawn to the scale $\frac{1}{48}$. Find the ratio of the area of the plan to the floor space it represents.

7. Compare the area of the surface of a tennis-ball $2\frac{1}{4}$ in. in diameter with the area of a baseball $2\frac{4}{5}$ in. in diameter.

8. John has a garden 14 ft. long. Harry has a garden similar in shape but 10 ft. long. What is the ratio of the size of John's to Harry's garden?

9. Mary tethered her pony with a rope 12 ft. long. How many times as much area could the pony have grazed over if the rope had been 15 ft.? 18 ft.?

10. If the earth's radius is taken as 4000 mi. and that of the moon as 1100 mi., what is the ratio of their areas?

11. The diameter of the sun is about 864,500 mi. Compare the area of the sun with that of the earth.

12. Compare the volumes of two cubes whose edges are 8 in. and 6 in.; 10 in. and 15 in.

13. The dimensions of one cube are 1.5 times that of another. Compare their areas.

14. The radii of two similar cones are 3.2 in. and 1.8 in. Compare their volumes and the areas of their surfaces.

15. What is the ratio of the amount of marble in a block 8 in. thick to a similar block 6 in. thick?

16. What is the ratio of the volumes of the two balls mentioned in Ex. 7?

17. A quart milk bottle is 9 in. high. What will be the height of a pint bottle of the same shape?

18. If a pail 6 in. high holds a quart, how much will a similar pail 8 in. high hold? 12 in. high?

19. If there are 60 sq. in. in the curved surface of a tin baking-powder can that is 6 in. high, how much tin will be needed for the curved surface of a similar can $3\frac{1}{2}$ in. high?

20. The ice compartment of a refrigerator which holds 100 lb. of ice is 14 in. wide. The ice compartment in another refrigerator is similar and 10 in. wide. About how many pounds of ice will the latter hold?

177. Direct and Inverse Proportion.—So far we have studied only quantities that are **directly proportional.** An increase in either of the two quantities also increases the other. As the radius of a circle increases, the circumference increases.

Quantities may also be **inversely proportional.** Suppose that the base and the altitude of a triangle change but that its area remains the same. Then, as the base increases, the altitude must **decrease in the same ratio.** Other illustrations of quantities inversely proportional are: the time that a pound of butter will last and the amount that is eaten daily; the number of articles and the price of each that can be bought for a certain sum of money; the thickness with which paint is spread on and the surface that can be painted with a certain amount of paint. Explain each statement.

Equation (1) shows two quantities, C and R, directly proportional. If a triangle has a constant area, 12, with base, b, and altitude, a, as in equation (2), then a and b are **inversely proportional.**

$$C = \tfrac{44}{7}R \qquad (1)$$
$$12 = \tfrac{1}{2}\,ab \qquad (2)$$

EXERCISES

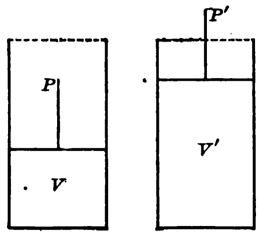
1. Show that the number of articles that can be bought for \$ 5 and the price of each are inversely proportional.

2. Show that the total price of oranges at 10 ¢ each is directly proportional to the number bought.

3. Explain that daily wages should be inversely proportional to the time required to do a certain piece of work.

4. Show that the number of strawberries in a box is *inversely* proportional to their size.

178. Application of Indirect Proportion.—If pressure be exerted upon a piston inserted into a cylinder filled with a gas, the volume of the gas will become smaller. If the pressure be doubled, the volume will be only one-half; if the pressure be made three times as great, the volume will be only one-third; and so on. The pressure and the volume thus vary inversely. This is known as Boyle's Law and is stated in symbols,



$$\frac{P}{P'} = \frac{V'}{V}, \qquad (1)$$

or

$$PV = P'V'. \qquad (2)$$

The 6 gal. of gas contained in a cylinder has a pressure of 15 lb. per square inch exerted upon it. What will be the volume if the pressure be increased to 18 lb. per square inch?

From (2) $\qquad 18V = 15 \times 6 = 90 \qquad (3)$

and $\qquad V = 90 \div 18 = 5$ gal. $\qquad (4)$

EXERCISES

1. The 6 gal. of gas contained in a cylinder has a pressure of 15 lb. per square inch exerted upon it. What will be the volume of the gas if the pressure is changed to 50 lb. per square inch? to 20 lb. per square inch?

2. A cylinder contains 8 gal. of a gas at a pressure of 20 lb. per square inch. How much pressure must be exerted upon it in order that the volume may be 6 gal.? 4 gal.? 12 gal.?

3. Graph the equation, $PV = 180$.

4. From the graph find V when P is 12; 20. Also find P when V is 6; 18; 24; 15.

XI

LOGARITHMS

179. Review of Exponents.—Review the principles involved in the following exercises and apply correctly in their solution:

$a^m \times a^n = a^{m+n}$	$10^2 \times 10^3 = 10^5$	How?
$a^m \div a^n = a^{m-n}$	$8^5 \div 8^3 = 8^2$	How?
$(a^m)^n = a^{mn}$	$(7^3)^2 = 7^6$	How?
$(a^m)^{\frac{1}{n}} = a^{\frac{m}{n}}$	$(10^3)^{\frac{1}{2}} = 10^{\frac{3}{2}}$	How?

EXERCISES

1. $a^3 \times a^2 = ?$
2. $s^r \times s^t = ?$
3. $p^{2a} \times p^{3a} = ?$
4. $2^2 \times 2^3 = ?$
5. $3^5 \times 3^2 = ?$
6. $b^{10} \div b^3 = ?$
7. $3^{10} \div 3^7 = ?$
8. $10^a \times 10^b = ?$
9. $10^2 \times 10^3 = ?$
10. $10^6 \div 10^3 = ?$
11. $10^{0.3010} \times 10^{0.4771} = ?$
12. $10^{0.6990} \times 10^{0.9030} = ?$
13. $(c^2)^3 = ?$
14. $(b^5)^2 = ?$
15. $(g^2)^4 = ?$
16. $(3^2)^3 = ?$
17. $(10^2)^3 = ?$
18. $(10^{0.3010})^3 = ?$
19. $(10^{0.4771})^2 = ?$
20. $(10^{1.5})^2 = ?$
21. $(10^{0.3010})^2 = ?$
22. $(10^{1.6020})^2 = ?$
23. $10^{0.4771} \div 10^{0.3010} = ?$
24. $10^{0.9030} \div 10^{0.6020} = ?$
25. $(b^2)^{\frac{1}{3}} = ?$
26. $(g^3)^{\frac{1}{6}} = ?$
27. $(3^3)^2 = ?$
28. $(k^4)^{\frac{1}{2}} = ?$
29. $(10^2)^{\frac{1}{4}} = ?$
30. $(K^{0.6020})^{\frac{1}{2}} = ?$
31. $(10^{0.6020})^{\frac{1}{2}} = ?$
32. $(10^{0.50})^{\frac{3}{2}} = ?$
33. $(10^{0.9030})^{\frac{3}{2}} = ?$
34. $(10^{0.3010})^{\frac{2}{5}} = ?$
35. $(10^{5.7781})^2 = ?$
36. $(10^{2.3010})^{\frac{2}{5}} = ?$

Hold a number contest on operations with exponents.

State and illustrate the following general laws of exponents in operations with a quantity, as q, raised to different exponents, as q^8 and q^3:

37. Multiplication of two or more such numbers.

38. Division of one such number by another.

39. Multiplication of some and division by other numbers.

40. Raising a quantity to a power.

41. Extracting a root of a quantity.

180. Value of a^0.—We have seen that $a^n \div a^n = a^{n-n} = a^0$. But we have also seen that

$$a^n \div a^n = \frac{a^n}{a^n} = 1.$$

Hence, since $a^n \div a^n$ equals both a^0 and 1; we know that

$$a^0 = 1,$$

for any value of a. Carry out the same operations for 5^0; for $(89)^0$.

181. Logarithms.—We have learned that $10^2 = 100$, that $10^3 = 1{,}000$, and so on. From this $10^2 \times 10^3 = 10^5 = 100{,}000$. Exponents used in this way are called **logarithms.**

Thus, the logarithm of 100 is 2; the logarithm of 1,000 is 3; and so on. The 10 is called the **base** of the logarithm system. Any number can be used as base of a logarithm system, but 10 will be used, as it is very much simpler than any other number. The reason for this will come out later.

Since $10^3 = 1{,}000$, what is the logarithm of 1,000? Similarly, what is the logarithm of 100? of 10? of 10,000? Since $10^0 = 1$, what is the logarithm of 1?

What is the number that has the logarithm 2? the logarithm 4? the logarithm 1? the logarithm 5?

182. Further Logarithms.—Logarithms of the common usable numbers have been computed and arranged in tables. A few of these are found to the right. Thus,

$$2 = 10^{0.3010} \text{ and } 3 = 10^{0.4771}.$$

In this same way express 5 and 8.

log 1	=	.0000
log 2	=	.3010
log 3	=	.4771
log 4	=	.6020
log 5	=	.6990
log 6	=	.7781
log 7	=	.8451
log 8	=	.9030
log 9	=	.9542
log 10	=	1.0000
log 100	=	2.0000

183. Multiplication and Division by Logarithms.—To find $4 \times 9 \div 6 \div 2$, we express each as 10 raised to a power; thus,

$$4 \times 9 \div 6 \div 2 =$$
$$10^{0.6020} \times 10^{0.9542} \div 10^{0.7781} \div 10^{0.3010} =$$
$$10^{0.4771}. \text{ How?}$$

The table shows that 0.4771 is the logarithm of 3. Hence, $4 \times 9 \div 6 \div 2 = 3$. Logarithms greatly simplify multiplication and division with large numbers. Since logarithms are exponents, formulate a law for using logarithms in multiplication and division.

EXERCISES

1. From the above table find the logarithms of the following: 3, 5, 9, 2, 7, 1.

2. From the above table find the numbers whose logarithms are: 0.8451, 0.3010, 2.000, 0.6990.

Carry out the following by the use of logarithms:

3. 2×3	8. $5 \times 4 \div 2$	13. $5 \times 4 \div 8 \times 2$
4. $6 \div 3$	9. $9 \div 6 \times 2$	14. $5 \times 4 \times 5$
5. 2×4	10. $8 \div 4 \times 5$	15. $5 \times 8 \div 4$
6. $6 \times 3 \div 9$	11. $3 \div 6 \times 4$	16. $6 \times 4 \div 8$
7. $4 \times 3 \div 2$	12. $4 \times 5 \div 8 \times 2$	17. $100 \div 5 \div 4$

184. Logarithms of Larger Numbers.—We know that $10 = 10^1$, $100 = 10^2$, $1{,}000 = 10^3$, etc. That is, log 10 = 1, log 100 = 2, log 1,000 = 3, etc.

Also, we have had $10 \times 2 = 10^1 \times 10^{0.3010} = 10^{1.3010}$.
But $10 \times 2 = 20$, hence $10^{1.3010} = 20$, or log 20 = 1.3010.
Similarly, $10 \times 5 = 10^1 \times 10^{0.6990} = 10^{1.6990}$. Log 50 = ?
Again, $100 \times 2 = 10^2 \times 10^{0.3010} = 10^{2.3010}$, or log 200 = 2.3010.
Similarly, $100 \times 7 = 10^2 \times 10^{0.8451} = 10^?$ and log 700 = ?

$$800 \div 100 = 10^{2.9030} \div 10^2 = 10^{2.9030-2} = 10^{0.9030} = ?$$
$$400 \div 100 = 10^{2.6020} \div 10^2 = 10^? = ?$$

Hence, to multiply by powers of 10, add 1 to the logarithm for each power of 10. To divide by powers of 10, subtract 1 from the logarithm for each power of 10. Note that the decimal part of a logarithm depends only upon the digits used and not upon the position of the decimal point in the number. The whole number in a logarithm depends only upon the position of the decimal point in the number. Thus, 40, 4, 400, 4,000, etc., all have the same decimal part in their logarithms. What is this? 300, 900, 100, etc., all have the same whole number in their logarithm. Show that this whole number is 2.

The whole number of the logarithm of any number is 1 less than the number of digits in the number to the left of the decimal point.

EXERCISES

1. What is the logarithm of 5? of 50? of 500? of 5,000?
2. What is the logarithm of 70? of 7? of 7,000? of 700?
3. What number has the logarithm 1.3010? 3.4771? 2.6990?
4. What number has the logarithm 2.3010? 1.4771? 3.6990?

185. Multiplication and Division Extended.—

$$80 \div 600 \times 30 = 10^{1.9030} \div 10^{2.7781} \times 10^{1.4771}$$
$$= 10^{1.9030 - 2.7781 + 1.4771} = 10^{0.6020} = ?$$

Make up a law on the use of logarithms in multiplication and division. Show that it conforms to the law of exponents.

186. Raising to Powers and Extracting Roots.—Logarithms can be used in raising numbers to powers or in extracting their roots. Thus,

$$\sqrt{9} = 9^{\frac{1}{2}} = (10^{0.9542})^{\frac{1}{2}} = 10^{0.4771}. \text{ How?}$$

From the table on page 208 we find the number whose logarithm is 0.4771. What is this number? In practice merely find the logarithm and divide by the index of the root. Thus, to find $\sqrt[3]{8}$, refer to the table and find that the logarithm of 8 is 0.9030. This divided by 3 gives 0.3010, the logarithm of 2.

To raise a number to a power, find its logarithm and multiply by the index of the power. Thus, to find $(20)^2$, refer to the table for the logarithm of 20, which is 1.3010. How? This is multiplied by 2, giving 2.6020 for the logarithm of the number sought. Show that this is 400.

EXERCISES

Carry out the following with logarithms:

1. $300 \div 60 \times 20$
2. $600 \div 80 \times 4$
3. $9 \times 400 \div 30 \div 2$
4. $20 \div 600 \times 90$
5. $400 \times 30 \div 6$
6. $50 \times 400 \div 20$
7. $\sqrt{400}$
8. $(30)^2$
9. $4 \times 600 \div 80$
10. $50 \times 800 \div 40$
11. $8 \times 5000 \div 40$
12. $(200)^3$
13. $\sqrt[3]{8000}$
14. $(2000)^2$
15. $\sqrt{40000}$
16. $5^2 \times 8 \div 40$

187. Tables of Logarithms.—Tables of logarithms are found on pages 228 and 229. These are only the decimal parts, just as those on page 208. In the tables the logarithms of 643, 64.3, 6.43, and so on, are hence the same. Why? What is the whole number of the logarithm of each of these numbers? To find the decimal part of the logarithm of 643, look down the left column until 64 is reached, then look to the right to the column having 3 at the top, where 8082 is found. Hence, the logarithm of 643 is 2.8082. How? The logarithm of 64.3 is 1.8082; of 64300 is 4.8082. How? Since the decimal part of the logarithm of 27 and 270 is the same, to find the logarithm of 27, look down the left column to 27 and under column 0, where 4314 is found. Hence, the logarithm of 27 is 1.4314. What is the logarithm of 270? of 2700? of 2.7? Again, since the decimal part of the logarithm of 4 and 400 is the same, to find the logarithm of 4, look down the left column to 40 and then, under column 0, will be found 6021. Hence, the logarithm of 4 is 0.6021, while the logarithm of 40 is 1.6021. What is the logarithm of 400? of 4000?

EXERCISES

Find the logarithms of the following numbers:

1. 275; 27.5; 2750; 2.75
2. 457; 4.57; 4570; 45.7
3. 715; 7.15; 71500; 71.5
4. 405; 40.5; 4.05; 40500
5. 203; 2.03; 20300; 20.3
6. 360; 36; 3.6; 36000
7. 700; 70; 7; 70000
8. 607; 34.7; 9.06; 30.5
9. 314; 20.8; 304; 2.8
10. 34; 4.5; 6.07; 8.34

11. Suggest a list of five numbers having one, two, or three digits, and find their logarithms.

188. Numbers of Logarithms.—Reverse the process to find the number having a given logarithm. Thus, to find the number whose logarithm is 3.5465, look until 5465 is found in the table. This is opposite 35 and in the column marked 2 at the top. Hence, 0.5465 is the decimal part of the logarithm of 352. The whole number 3 of the logarithm makes the number 3520. How?

EXERCISES

Find the numbers having the following logarithms:

1. 1.5729; 2.5729; 0.5729
2. 0.5132; 3.5132; 1.5132
3. 1.0212; 0.0212; 3.0212
4. 0.9445; 2.9445; 1.9445
5. 1.9850; 0.9850; 3.9850
6. 2.0792; 0.0792; 1.0792

Carry out the following computations:

7. 125×328
8. 375×456
9. 425×840
10. $728 \div 182$

189. Logarithms not Found in the Tables.—Find in the tables the number having 0.5378 as the decimal part of its logarithm. Also find the number having 0.5391 as the decimal part of its logarithm. In carrying out a computation it may be necessary to find the number which has 0.5381 as the decimal part of its logarithm. As 5381 is not found in the tables, we use for the present the number with the logarithm nearest to 5381, or 345.

EXERCISES

Find the numbers having the following logarithms:

1. 0.4380; 2.4380; 1.4380
2. 1.7862; 0.7862; 3.7862
3. 1.7216; 3.7216; 0.7216
4. 1.9458; 0.9458; 2.9458
5. 0.9302; 3.9302; 1.9302
6. 1.6410; 0.6410; 3.6410

Carry out the following computations:

7. 354×278
8. 716×391
9. $706 \div 234$
10. $546 \div 358$

190. Computations with Logarithms.—Find $\frac{1260 \times 748}{396 \times 119}$.

$$\begin{array}{llll}
\log 1260 = 3.1004 & \text{How?} & \log 396 = 2.5977 & \text{How?} \\
\quad\text{"}\quad 748 = \underline{2.8739} & \text{How?} & \quad\text{"}\quad 119 = \underline{2.0755} & \text{How?} \\
\qquad\qquad 5.9743 & & \qquad\qquad 4.6732 &
\end{array}$$

$$\text{By subtraction,} \quad \begin{array}{r} 5.9743 \\ \underline{4.6732} \\ 1.3011 \end{array} \quad \text{Why subtract?}$$

The number nearest to 3011 found in the tables is 3010 and this is the decimal part of the logarithm of 2. The whole number of the logarithm tells us that there are two digits in the number. Hence,

$$\frac{1260 \times 748}{396 \times 119} = 20.$$

EXERCISES

Use logarithms in carrying out the following to the nearest number:

1. $\frac{119 \times 87}{287}$
2. $\frac{56 \times 125}{175}$
3. $\frac{47 \times 4850}{235}$
4. $\frac{45 \times 63 \times 74}{333 \times 90}$
5. $\frac{38 \times 968}{209 \times 44}$
6. $\frac{26 \times 144 \times 72}{324 \times 96}$
7. $\frac{3090 \times 84}{412 \times 30}$
8. $\frac{1020 \times 858}{68 \times 39 \times 330}$
9. $\frac{144 \times 278 \times 15}{108 \times 40}$
10. $\frac{135 \times 3000 \times 26}{150 \times 270 \times 65}$
11. $\frac{145 \times 3.5}{20.3}$
12. $\frac{48.5 \times 9.5}{2.35}$
13. $\frac{3.85 \times 8.47}{8.32}$
14. $\frac{28.4 \times 35.8}{17.1 \times 4.56}$

Find $\sqrt[5]{\dfrac{34.5 \times 6.45}{12.8}}$

$$\begin{array}{rrl} \textit{log } 34.5 = & 1.5378 & \\ + \text{ “ } 6.45 = & \underline{0.8096} & \text{Why add?} \\ & 2.3474 & \\ - \text{ “ } 12.8 = & \underline{1.1072} & \text{Why subtract?} \\ & 5)\overline{1.2402} & \\ & .2480 & \text{Why divide?} \end{array}$$

Looking in the tables, we find that 0.2480 is the logarithm of 1.77, which is the result sought.

It is a good plan to write down first the general plan in such a computation as that above. This is the part shown in italics. The logarithms are next found, then the number work is carried out, and finally the number corresponding to the last logarithm is found.

15. $\sqrt{24.5 \times 314}$

16. $\sqrt[4]{\dfrac{70.5 \times 4.58}{65.8}}$

17. $\sqrt[5]{\dfrac{4560}{4.56 \times 175}}$

18. $\sqrt[3]{2.56 \times 45.6}$

19. $\sqrt[6]{\dfrac{3.05 \times 96.8}{73.8}}$

20. $\sqrt[7]{\dfrac{4500}{15.6 \times 8.25}}$

Use logarithms in the following problems:

21. Write the formula for the area of a triangle. Find the area of the triangle whose base is 364 ft. and altitude 198 ft.

22. What is the area of the triangle whose base is 62.5 in. and altitude 54.8 in.?

23. What is the area of a triangle with a base 52.6 m. long and an altitude 84.5 m. long?

24. Express the edge of a cube in terms of its volume. Find the edge of a cube that has a volume of 125 cu. in.; of 452 cu. ft.; of 678 cm.^{3}; of 1435 m.^{3}

Hold a number contest on using logarithms.

191. Logarithms of Decimals.—Suppose that the logarithm of 0.00375 is needed. On page 228 the logarithm of 3.75 is found to be 0.5740. But $0.00375 = 3.75 \div 10^3$, or $10^{0.5740} \div 10^3 = 10^{0.5740-3}$. Hence, the logarithm of 0.00375 is $0.5740 - 3$. Always write negative logarithms in this manner: the positive decimal minus the whole number.

EXERCISES

Find the logarithms of the following from the tables on pages 228 and 229:

1. 0.0765	6. 0.0597	11. 0.234
2. 0.0119	7. 0.00874	12. 0.000175
3. 0.00045	8. 0.0507	13. 0.0107
4. 0.00827	9. 0.0484	14. 0.0019
5. 0.357	10. 0.0706	15. 0.0508

192. Finding Numbers Having Negative Logarithms.—Find the number whose logarithm is $0.4969 - 4$. In the table is found 0.4969 opposite 31 and in the column with 4 at the top. Hence, 0.4969 is the logarithm of 3.14, but since the logarithm is $0.4969 - 4$ this number is divided by 10^4, which gives 0.000314 as the number sought.

EXERCISES

Find the numbers having the following logarithms:

1. $0.4843 - 2$	6. 1.9004	11. 3.7846
2. $0.5635 - 1$	7. $0.5224 - 4$	12. $0.8526 - 2$
3. 3.5635	8. $0.5490 - 1$	13. $0.7845 - 3$
4. $0.9956 - 3$	9. $0.8762 - 1$	14. 3.8527
5. $0.9004 - 2$	10. 3.8762	15. $0.5773 - 1$

Hold a number contest, using whole numbers and decimals.

193. Operations with Negative Logarithms.—Find

$$\frac{34.7 \times 2.35}{123}.$$

$$\begin{array}{rrl}
 & \log 34.7 = & 1.5403 \\
 & \text{"}\quad 2.35 = & \underline{0.3711} \\
 & & 2.9114 - 1 \\
- & \text{"}\quad 123 = & \underline{2.0899} \\
 & & .8215 - 1
\end{array}$$

Since it is impossible to take 2.0899 from 1.9114 we change the latter to $2.9114 - 1$ by adding and subtracting 1. Subtraction gives $0.8215 - 1$ as the logarithm of the number sought. 0.8215 is the logarithm of 6.63, hence $0.8215 - 1$ is the logarithm of 0.663.

Find $(.745)^{\frac{2}{3}}$. The logarithm of 0.745 is multiplied by $\frac{2}{3}$ and the number having this logarithm found. Log $0.745 = 0.8722 - 1$. How? Multiplying by 2 gives $1.7444 - 2$. Since the negative whole number cannot be divided by 3 without giving a remainder, 1 is added and subtracted, giving $2.7444 - 3$, which, divided by 3, gives $0.9148 - 1$. 8.22 has a logarithm of 0.9149; hence, 8.22 is taken as the number with a logarithm 0.9148. Therefore 0.822 is the number sought. How?

EXERCISES

Carry out the following by the use of logarithms:

1. $\frac{60.8}{324}$
2. $\frac{92.5}{897}$
3. $(24.5)^{\frac{2}{3}}$
4. $\frac{81.5}{678}$
5. $\frac{45.7}{149}$
6. $(5.65)^{\frac{1}{5}}$
7. $\frac{7.08}{36.5}$
8. $\frac{34.1}{216}$
9. $(0.745)^{\frac{3}{4}}$
10. $\frac{56.8 \times 12.6}{4450}$
11. $\frac{6.95 \times 4.72}{97.4}$
12. $\frac{284}{135 \times 85.7}$
13. $\frac{62.7}{9.45 \times 12.3}$
14. $\frac{23.4 \times 86}{35 \times 96.4}$
15. $\frac{4.65 \times 32.8}{506}$

16. $\dfrac{204 \times 325}{854000}$

17. $\sqrt{\dfrac{24.5 \times 36.5}{2350}}$

18. $\sqrt[4]{\dfrac{89.6}{32.1 \times 24}}$

19. $\sqrt[6]{\dfrac{25 \times 214}{117 \times 318}}$

Use logarithms in solving the following problems:

20. The formula for the circumference of a circle is

$$C = \frac{44}{7}r.$$

Find the circumference of the circle whose radius is 23.5 in.; 8.45 ft.; 635 in.

21. Find the circumference of the circle whose radius is 13.5 m.; 6.45 Dm.; 325 dm.

22. Solve the above formula giving the radius in terms of the circumference. Find the radius of the circle whose circumference is 45.6 ft.; 18.6 yd.; 345 in.

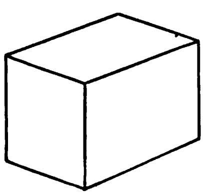
23. Find the radius of the circle whose circumference is 65.8 m.; 2450 dm.; 3815 Dm.

24. Engineers use the formula,

$$D = \frac{W \times L}{48 \times E \times I}.$$

Find the value of D when $W = 4.25$; $L = 23.6$; $E = 18{,}500$; $I = 0.35$.

25. Find the value of D when $W = 6.5$; $L = 18.6$; $E = 20{,}000$; $I = 0.65$.



26. Write the formula for the volume of a rectangular box. Find the volume of a rectangular box with the following dimensions: 45.6 in. × 32.8 in. × 70.5 in.

27. Find the number of bushels of wheat a bin will hold having the dimensions 8.5 ft. × 15.6 ft. × 26.5 ft.

Hold a number contest on using logarithms.

194. Logarithms of Numbers with Several Digits.—The decimal part of the logarithm of 3670 is 0.5647, while the decimal part of the logarithm of 3680 is 0.5658. Hence, the logarithm of 3675 will be midway between these, or 0.5647 + .0006 = 0.5653. What is the logarithm of 36.75? of 3.675? of 0.3675? of 367500?

To find the logarithm of 3674, first find the difference between the logarithms of 3670 and 3680. Multiply this by .4 and add the resulting product to the logarithm of 3670. Hence, the decimal part of the logarithm of 3674 is 0.5647 + 0.0004 = 0.5651. Similarly, the decimal part of the logarithm of 3678 is $0.5647 + 0.0011 \times .8$, or 0.5656. What is the logarithm of 3678? of 36.78? Find the decimal part of the logarithm of 3673; of 3676.

Always find the decimal part of the logarithm first and then the whole number from the location of the decimal point in the number.

EXERCISES

Find the logarithms of the following:

1.	3673	**11.**	4562	**21.**	45.06
2.	3679	**12.**	3056	**22.**	8.034
3.	3025	**13.**	2507	**23.**	120.5
4.	8425	**14.**	3045	**24.**	25.05
5.	19.34	**15.**	730.5	**25.**	61.32
6.	8.325	**16.**	200.5	**26.**	8.005
7.	34.54	**17.**	19.45	**27.**	17.05
8.	61.06	**18.**	73.06	**28.**	8.047
9.	16.08	**19.**	5.607	**29.**	304.7
10.	7.054	**20.**	13.45	**30.**	60.58

195. Logarithms not Found in the Tables.—Find the number which has 0.4221 as the decimal part of its logarithm. The decimal part of the logarithm of 2640 is 0.4216 and of 2650 is 0.4232. The decimal part of the logarithm of 2650 is larger than that of 2640 by $0.4232 - 0.4216 = 0.0016$. Also 0.4221 is larger than 0.4216 by $0.4221 - 0.4216 = 0.0005$. Hence, the given logarithm 0.4221 is $\frac{0.0005}{0.0016}$, or 0.3, of the difference between the two logarithms 0.4232 and 0.4216. Hence, the number corresponding to 0.4221 is 0.3 of the difference between 2640 and 2650, which is 2643. Hence, the number that has 0.4221 as the decimal part of its logarithm is 2643.

Similarly, the number whose decimal part of the logarithm is 0.4227 is $\frac{0.0011}{0.0016}$, or merely $\frac{11}{16}$, of the difference between 2640 and 2650. As $\frac{11}{16} = .7$, the number having 0.4227 as the decimal part of its logarithm is 2647. What number has the logarithm 1.4227 ?

EXERCISES

Find the numbers whose logarithms are the following:

1.	3.1725	**6.**	2.6175	**11.**	$0.6175 - 3$
2.	1.2364	**7.**	0.6842	**12.**	$0.8770 - 1$
3.	0.3475	**8.**	$0.6842 - 1$	**13.**	0.9289
4.	1.2090	**9.**	$0.1842 - 2$	**14.**	4.9793
5.	2.3518	**10.**	$0.3475 - 2$	**15.**	3.9725

Carry out the following by the use of logarithms:

16.	2.75×4.75	**21.**	32.56×5.034	**26.**	135^2
17.	4.23×32.5	**22.**	3141×235	**27.**	$\sqrt{439}$
18.	$34.5 \div 6.24$	**23.**	35.37×4.557	**28.**	347^2
19.	4235×23.41	**24.**	$31.67 \div 6.725$	**29.**	$(519)^{\frac{1}{5}}$
20.	$7419 \div 156$	**25.**	$42.74 \div 5.37$	**30.**	413^7

31. $\dfrac{54.2 \times 632 \times 0.083}{3.5 \times 9.2}$

32. $\dfrac{465 \times 2.38 \times 53.8}{832 \times 45.8}$

33. $\dfrac{18.3 \times 27 \times 71.3}{40.5 \times 307}$

34. $\dfrac{3527 \times 6405}{7142 \times 115}$

35. $\dfrac{34.57 \times 8.035}{47.28}$

36. $\dfrac{8.305 \times 32.17}{602 \times 31.56}$

37. $\dfrac{32.05 \times 153 \times 3.014}{41.05 \times 50.34}$

38. $\dfrac{182.5 \times 3.046 \times 45.12}{30.37 \times 129.6}$

39. $\dfrac{3.05 \times 29.71 \times 56.07}{45.12 \times 82.19}$

40. If a sum of money, P, is placed at interest compounded semiannually at r per cent for n years, the accumulation is expressed by the formula,

$$A = P\left(1 + \frac{r}{2}\right)^{2n}.$$

Find the amount of a Baby Bond at \$ 4.12 from Jan. 1, 1918, to Jan. 1, 1923, interest compounded semiannually at 4 %. $P = \$\,4.12$, $r = 0.04$, $n = 5$. Hence,

$$A = 4.12(1 + 0.02)^{10} = 4.12(1.02)^{10}.$$

Log 4.12 = 0.6149; log 1.02 = 0.0086; raised to the power 10 the logarithm is multiplied by 10 when we get .0860 for the logarithm of $(1.02)^{10}$. Adding the logarithms gives $0.6149 + 0.0860 = 0.7009$. The number which has this decimal part of the logarithm is nearly \$ 5.02. As the amount paid by the government for these bonds when due is \$ 5, the interest is very nearly 4 %.

41. Tom received \$ 10 on his tenth birthday. He placed it in the Building and Loan Association which pays 5 % interest compounded semiannually. What was the amount at his majority, 21 years of age?

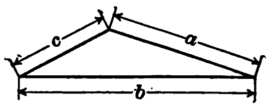
42. Find the accumulation of \$ 150 at $3\frac{1}{2}$ % for 4 yr., interest compounded semiannually.

43. Find the amount of \$ 325 at 4 % for 6 yr., interest compounded semiannually.

44. Which is the better investment and how much, \$ 100 at simple interest for 10 yr. at 5 %, or \$100 for 10 yr. at 4 % compounded semiannually?

45. Find how much \$ 100 will amount to at 5 % interest compounded semiannually for 25 yr.; for 50 yr.; for 75 yr.; for 100 yr.

It is proved in geometry that for any triangle



$$A = [s(s-a)(s-b)(s-c)]^{\frac{1}{2}}$$

in which a, b, and c are the lengths of the sides of the triangle and $s = \frac{1}{2}(a+b+c)$.

46. Find the area of the triangle whose sides are 246 ft., 805 ft., and 657 ft.

47. Find the area of the triangle whose sides are 45.32 ft., 117.6 ft., and 89.34 ft.

48. Find the area of the triangle whose sides are 57.04 ft., 43.14 ft., and 61.08 ft.

The number of gallons a cylindrical tank or pipe can hold is given by the formula,

$$G = 0.0034hd^2,$$

in which both h and d are given in inches.

49. Find the capacity in gallons of a cylindrical tank 3 ft. deep and 18 in. in diameter.

50. Find the capacity in gallons of a pipe 19.5 ft. high and with a diameter of 2.4 in.

Hold a number contest on using logarithms.

REFERENCE TABLES

ENGLISH SYSTEM

Length

12 inches (in.) = 1 foot (ft.).
3 feet = 1 yard (yd.).
$5\frac{1}{2}$ yards, or $16\frac{1}{2}$ feet = 1 rod (rd.).
320 rods, or 5280 feet = 1 mile (mi.).
5 ft. 3 in. may be written $5'\ 3''$.

SURVEYORS' TABLE OF LENGTH

7.92 inches = 1 link (li.).
100 links = 4 rods = 1 chain (ch.).
80 chains = 5280 ft. = 1 mile.

ADDITIONAL UNITS OF LENGTH

4 inches = 1 hand.
6 feet = 1 fathom.
100 fathoms = 1 cable length.
1.15 common miles = 1 knot (nautical mile).

Square Measure

144 square inches (sq. in.) = 1 square foot (sq. ft.).
9 square feet = 1 square yard (sq. yd.).
$30\frac{1}{4}$ square yards = 1 square rod (sq. rd.).
160 square rods = 1 acre (A.).
640 acres = 1 square mile (sq. mi.).

SURVEYORS' TABLE OF SQUARE MEASURE

16 square rods (sq. rd.) = 1 square chain (sq. ch.).
10 square chains = 1 acre (A.).
640 acres = 1 square mile (sq. mi.).
36 square miles = 1 township (tp.).

Cubic Measure

1728 cubic inches (cu. in.) = 1 cubic foot (cu. ft.).
27 cubic feet = 1 cubic yard (cu. yd.).
128 cubic feet = 1 cord (cd.).
$24\frac{3}{4}$ cubic feet = 1 perch (stone, etc.).
A cubic yard is called a load.

Weight

AVOIRDUPOIS

16 ounces (oz.) = 1 pound (lb.).
100 pounds = 1 hundredweight (cwt.).
2000 pounds = 1 ton (T.).
112 pounds = 1 long hundredweight.
2240 pounds = 1 long ton.
Long ton is used in U. S. Custom-house and at mines.
1 ton soft coal occupies about 35 cubic feet.
1 ton hard coal occupies about 28 cubic feet.

Liquid Measure

4 gills (gi.) = 1 pint (pt.).
2 pints = 1 quart (qt.).
4 quarts = 1 gallon (gal.).
231 cubic inches = 1 gallon.
31.5 gallons = 1 barrel (bbl.) (varies).
2 barrels = 1 hogshead (varies).
1 pint = 16 fluid ounces (apothecaries').
57.75 cubic inches = 1 liquid quart.
$\frac{1}{2}$ pint = 1 measuring cup.
1 cubic foot of water weighs nearly $62\frac{1}{2}$ pounds.
1 gallon of water weighs nearly $8\frac{1}{3}$ pounds.
1 cubic foot of water equals about $7\frac{1}{2}$ gallons.

Dry Measure

2 pints (pt.) = 1 quart (qt.).
8 quarts = 1 peck (pk.).
4 pecks = 1 bushel (bu.).
2150.4 cubic inches = 1 bushel.
1 stricken bushel = $1\frac{1}{4}$ cubic feet (nearly).
1 heaped bushel = $1\frac{1}{2}$ cubic feet (nearly).
1 bushel ear corn = $2\frac{1}{2}$ cubic feet (nearly).

The Following Hold in Nearly All States

1 bushel of wheat weighs 60 pounds.
1 bushel of shelled corn weighs 56 pounds.
1 bushel of ear corn weighs 75 or 80 pounds in the fall and 70 pound later.
1 bushel of oats weighs 32 pounds.
1 bushel of rye weighs 56 pounds.
1 bushel of barley weighs 48 pounds.
1 bushel of potatoes weighs 60 pounds.
1 bushel of beans weighs 60 pounds.
1 bushel of peas weighs 60 pounds.
1 bushel of apples weighs 48 pounds.
1 bushel of clover seed weighs 60 pounds.
1 bushel of alfalfa seed weighs 60 pounds.
1 bushel of timothy seed weighs 45 pounds.
1 bushel of bran weighs 20 pounds.
1 bushel of soft coal weighs 80 pounds.
1 barrel of flour weighs 196 pounds.
1 barrel of pork or beef weighs 200 pounds.

Angles and Arcs

60 seconds (60″) = 1 minute (1′).
60 minutes = 1 degree (1°).
90 degrees = 1 right angle.
360 degrees = 1 circumference.

Time

60 seconds (sec.) = 1 minute (min.).
60 minutes = 1 hour (hr.).
24 hours = 1 day (da.).
7 days = 1 week (wk.).
12 months (mo.) = 1 year (yr.).
365 days = 1 common year.
366 days = 1 leap year.

United States Money

10 mills = 1 cent (ct. or ¢).
10 cents = 1 dime (d.).
10 dimes = 1 dollar ($).
10 dollars = 1 eagle (E.).

Counting

12 units	=	1 dozen (doz.).
12 dozen, or 144	=	1 gross (gr.).
12 gross, or 1728	=	1 great gross.
20 units	=	1 score.

The dozen is being replaced by 10 and the gross by 100.

500 sheets of paper are called a ream.

METRIC SYSTEM

Length

10 millimeters (mm.)	=	1 centimeter (cm.).
10 centimeters	=	1 decimeter (dm.).
10 decimeters	=	1 meter (m.).
10 meters	=	1 Dekameter (Dm.).
10 Dekameters	=	1 Hektometer (Hm.).
10 Hektometers	=	1 Kilometer (Km.).

Square Measure

100 square millimeters (mm^2.)	=	1 square centimeter (cm^2.).
100 square centimeters (cm^2.)	=	1 square decimeter (dm^2.).
100 square decimeters	=	1 square meter (m^2.).
100 square meters	=	1 square Dekameter (Dm^2.).
100 square Dekameters	=	1 square Hektometer (Hm^2.).
100 square Hektometers	=	1 square Kilometer (Km^2.).

A square Dekameter is called an **are.** As 100 square Dekameters equals 1 square Hektometer, a square Hektometer is called a **hektare** (ha.). These are the metric units of land measure.

Cubic Measure

1000 cubic millimeters (mm^3.)	=	1 cubic centimeter (cm^3.).
1000 cubic centimeters	=	1 cubic decimeter (dm^3.).
1000 cubic decimeters	=	1 cubic meter (m^3.).
1000 cubic meters	=	1 cubic Dekameter (Dm^3.).
1000 cubic Dekameters	=	1 cubic Hektometer (Hm^3.).
1000 cubic Hektometers	=	1 cubic Kilometer (Km^3.).

The cubic meter is used in measuring wood and is called the **stere** (st.).

Weight

10 milligrams (mg.) = 1 centigram (cg.).
10 centigrams = 1 decigram (dg.).
10 decigrams = 1 gram (g.).
10 grams = 1 Dekagram (Dg.).
10 Dekagrams = 1 Hektogram (Hg.).
10 Hektograms = 1 Kilogram (Kg.).
1000 Kilograms = 1 Metric ton (t.).

The gram is the weight of 1 cm^3. of water at a temperature of 4° C.

Capacity

10 milliliters (ml.) = 1 centiliter (cl.).
10 centiliters = 1 deciliter (dl.).
10 deciliters = 1 liter (l.).
10 liters = 1 Dekaliter (Dl.).
10 Dekaliters = 1 Hektoliter (Hl.).
10 Hektoliters = 1 Kiloliter (Kl.).

The liter is 1 dm^3.

Equivalents

A meter	= 39.37 in.	= $3\frac{1}{4}$ ft. (nearly).
A Kilometer	= .621 mi.	= .6 mi. (nearly).
A liter	= 1.056 qt. (liquid)	= 1 qt. (nearly).
A liter	= .908 qt. (dry)	= .9 qt. (nearly).
A Kilogram	= 2.204 lb.	= 2.2 lb. (nearly).
A hectare	= 2.47 A.	= $2\frac{1}{2}$ A. (nearly).

Formulas of Areas and Volumes

Area parallelogram	$= ab$.
Area triangle	$= \frac{1}{2}\, ab$.
Area trapezoid	$= \frac{1}{2}\, a(B + b)$.
Area circle	$= \pi\, R^2$.
Area ring	$= \pi\, (R^2 - r^2)$.
Circumference circle	$= 2\, \pi\, R$.
Lateral surface regular pyramid	$= \frac{1}{2}\, nls$.
Lateral surface cone	$= \pi\, Rs$.
Surface sphere	$= 4\, \pi\, R^2$.
Volume pyramid	$= \frac{1}{3}\, Bh$.
Volume cone	$= \frac{1}{3}\, \pi\, R^2h$.
Volume sphere	$= \frac{4}{3}\, \pi\, R^3$.

Table of Specific Gravities

Aluminum	2.58	Ice	0.92	Oak	0.8
Copper	8.9	Iron (cast)	7.4	Petroleum	0.88
Cork	0.24	Iron (wrought)	7.8	Pine	0.6
Gold	19.3	Mercury	13.6	Silver	10.53

TABLES OF LOGARITHMS

The following two pages contain tables of the decimal part of logarithms of numbers having three digits or less.

To find the logarithm of the decimal part of a number, as 356, look down the left-hand column until 35 is reached, then to the right to the column having 6 at the top, where 5514 is found as the logarithm of 356. The logarithm of 72 is the same as of 720. The logarithm of 8 is the same as of 800. To find the logarithm of 3564, find .4 of the difference between the logarithms of 357 and 356, then add this to the logarithm of 356. Hence, the logarithm of 3564 is $5514 + 5 = 5519$. See Arts. 187, 191, and 194.

To find the number having a logarithm that does not appear in the tables, first find the number whose logarithm is next smaller. Find the difference between this next smaller and the given logarithm. Also, find the difference between the next smaller and the next larger logarithm. Multiply the smaller difference by 100, divide by the larger difference, and annex the digits to the number with the next smaller logarithm. In finding the number whose logarithm is 3309, note that 3304 is next smaller and their difference is 5. The difference between 3304 and the next larger logarithm is 20. Also, $500 \div 20 = 25$, to be annexed to 214, the number having the logarithm 3304. This gives the required number 21425. See Arts. 188, 189, 192, and 195.

Use the straight edge of a card or paper to follow the line across the page, to avoid errors.

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	2677
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

RATIOS OF ANGLES

Explanation of Table of Ratios.—If a right triangle is drawn with one angle 40°, the other acute angle will be 50°. It is seen from this triangle that the sine of 40° equals the cosine of 50°. Similarly, the tangent of 40° equals the cotangent of 50°. Again, the sine of 50° equals the cosine of 40°. That is, any ratio equals the **co**-ratio of the **complementary** angle. The sine of 30° equals the cosine of 60°; the cotangent of 70° equals the tangent of 20°.

Tables of ratios of angles are usually made for the first 45° only. Ratios of an angle between 45° and 90° are found from the **co**-named ratios of the **complementary angle.** Thus, to find sin 63° we find cos 27°. Similarly, for tan 49° we find cot 41°; for cos 81° we find sin 9°. So, to find a ratio of an angle between 45° and 90°, first find its complement and then find the **co**-named ratio in the tables.

In using the tables on the opposite page, note that when the angle is found in the column to the **left** the names of the ratios are given at the **top.** To find tan 36°, begin at the top of the column marked **tan** and go **down** until opposite 36 in the column to the left, where is found 0.7265 as the value of tan 36°. When the angle is found in the column to the **right** the names of the ratios are given **below** the columns. To find sin 76°, begin at the bottom of the column marked **sin** and go **up** until opposite 76 in the column to the right, where is found 0.9703 as the value of sin 76°.

The tables on the opposite page give only the ratios of angles of whole degrees, which is sufficient for our purposes. In higher mathematics and in engineering much larger tables are used, which give ratios of angles for minutes and seconds of arc.

	sin	cos	tan	cot	
0°	0.0000	1.0000	0.0000	———	90°
1°	0.0175	0.9998	0.0175	57.2900	89°
2°	0.0349	0.9994	0.0349	28.6363	88°
3°	0.0523	0.9986	0.0524	19.0811	87°
4°	0.0698	0.9976	0.0699	14.3007	86°
5°	0.0872	0.9962	0.0875	11.4301	85°
6°	0.1045	0.9945	0.1051	9.5144	84°
7°	0.1219	0.9925	0.1228	8.1443	83°
8°	0.1392	0.9903	0.1405	7.1154	82°
9°	0.1564	0.9877	0.1584	6.3138	81°
10°	0.1736	0.9848	0.1763	5.6713	80°
11°	0.1908	0.9816	0.1944	5.1446	79°
12°	0.2079	0.9781	0.2126	4.7046	78°
13°	0.2250	0.9744	0.2309	4.3315	77°
14°	0.2419	0.9703	0.2493	4.0108	76°
15°	0.2588	0.9659	0.2679	3.7321	75°
16°	0.2756	0.9613	0.2867	3.4874	74°
17°	0.2924	0.9563	0.3057	3.2709	73°
18°	0.3090	0.9511	0.3249	3.0777	72°
19°	0.3256	0.9455	0.3443	2.9042	71°
20°	0.3420	0.9397	0.3640	2.7475	70°
21°	0.3584	0.9336	0.3839	2.6051	69°
22°	0.3746	0.9272	0.4040	2.4751	68°
23°	0.3907	0.9205	0.4245	2.3559	67°
24°	0.4067	0.9135	0.4452	2.2460	66°
25°	0.4226	0.9063	0.4663	2.1445	65°
26°	0.4384	0.8988	0.4877	2.0503	64°
27°	0.4540	0.8910	0.5095	1.9626	63°
28°	0.4695	0.8829	0.5317	1.8807	62°
29°	0.4848	0.8746	0.5543	1.8040	61°
30°	0.5000	0.8660	0.5774	1.7321	60°
31°	0.5150	0.8572	0.6009	1.6643	59°
32°	0.5299	0.8480	0.6249	1.6003	58°
33°	0.5446	0.8387	0.6494	1.5399	57°
34°	0.5592	0.8290	0.6745	1.4826	56°
35°	0.5736	0.8192	0.7002	1.4281	55°
36°	0.5878	0.8090	0.7265	1.3764	54°
37°	0.6018	0.7986	0.7536	1.3270	53°
38°	0.6157	0.7880	0.7813	1.2799	52°
39°	0.6293	0.7771	0.8098	1.2349	51°
40°	0.6428	0.7660	0.8391	1.1918	50°
41°	0.6561	0.7547	0.8693	1.1504	49°
42°	0.6691	0.7431	0.9004	1.1106	48°
43°	0.6820	0.7314	0.9325	1.0724	47°
44°	0.6947	0.7193	0.9657	1.0355	46°
45°	0.7071	0.7071	1.0000	1.0000	45°
	cos	sin	cot	tan	

TABLE OF SQUARE ROOTS

2	1.4142	26	5.0990	51	7.1414
3	1.7321	29	5.3852	53	7.2801
5	2.2361	30	5.4772	55	7.4162
6	2.4495	31	5.5678	57	7.5498
7	2.6458	33	5.7446	58	7.6158
10	3.1623	34	5.8310	59	7.6811
11	3.3166	35	5.9161	61	7.8102
13	3.6056	37	6.0828	62	7.8740
14	3.7417	38	6.1644	65	8.0623
15	3.8730	39	6.2450	66	8.1240
17	4.1231	41	6.4031	67	8.1854
19	4.3589	42	6.4807	68	8.2462
21	4.5826	43	6.5574	69	8.3066
22	4.6904	46	6.7823	70	8.3666
23	4.7958	47	6.8557	71	8.4261

$\pi = 3.1416$ 1.7725

TABLE OF CUBE ROOTS

2	1.2599	21	2.7589	42	3.4760
3	1.4422	22	2.8020	43	3.5034
4	1.5874	23	2.8439	44	3.5303
5	1.7100	25	2.9240	45	3.5569
6	1.8171	26	2.9625	46	3.5830
7	1.9129	28	3.0366	47	3.6088
9	2.0801	29	3.0723	49	3.6593
10	2.1544	30	3.1072	50	3.6840
11	2.2240	31	3.1414	51	3.7084
12	2.2894	33	3.2075	52	3.7325
13	2.3513	34	3.2396	53	3.7563
14	2.4101	35	3.2711	55	3.8030
15	2.4662	36	3.3019	57	3.8485
17	2.5713	37	3.3322	58	3.8709
18	2.6207	38	3.3620	59	3.8930
19	2.6684	39	3.3912	60	3.9149
20	2.7144	41	3.4482	61	3.9365

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